From Qualitative Absolute Order-of-Magnitude to the Extended Set of Hesitant Fuzzy Linguistic Term Sets

Jordi Montserrat-Adell* and **Mónica Sánchez** and **Francisco Javier Ruiz**
UPC-BarcelonaTech, Barcelona, Spain
{jordi.montserrat-adell,monica.sanchez,francisco.javier.ruiz}@upc.edu

Núria Agell
Esade - Universitat Ramon Llull, Barcelona, Spain
nuria.agell@esade.edu

Abstract

Hesitant fuzzy linguistic term sets were introduced to grasp the uncertainty existing in human reasoning. In this paper, inspired by absolute order-of-magnitude qualitative reasoning techniques, an extension of the set of hesitant fuzzy linguistic term sets is presented to capture differences between non-compatible preferences. In addition, an order relation and two closed operations over this set are also introduced to provide a lattice structure to the extended set of hesitant fuzzy linguistic term sets. Based on this lattice structure a distance between hesitant fuzzy linguistic term sets is defined.

Keywords: Linguistic modeling, Group decision making, Uncertainty and Fuzzy Reasoning, Hesitant fuzzy linguistic term sets.

Introduction

Techniques based on order-of-magnitude qualitative reasoning have provided theoretical models to deal with non-numeric variables (Agell et al. 2012; Forbus 1996; Travé-Massuyès and Dague 2003; Travé-Massuyès et al. 2005). One of the advantages of qualitative reasoning is its capability to tackle problems in such a way that the principle of relevance is preserved; that is to say, each variable involved in a real problem must be valued at the precision level required. Order-of-magnitude models are among the essential theoretical tools available for qualitative reasoning about real systems. They aim to capture order-of-magnitude commonsense inferences, as used by human beings in the real world.

In addition, different approaches involving linguistic assessments have been introduced in the fuzzy sets literature to deal with the impreciseness and uncertainty connate with human reasoning (Espinilla, Liu, and Martínez 2011; Herrera, Herrera-Viedma, and Martínez 2008; Herrera-Viedma, Herrera, and Chiclana 2002; Parreiras et al. 2010; Tang and Zheng 2006). Additionally, different extensions of fuzzy sets have been presented to give more realistic assessments when uncertainty increases (Deschrijver and Kerres 2003; Greenfield and Chiclana 2013; Rodríguez, Martínez, and Herrera 2012). To describe human reasoning with different levels of precision similarly to absolute order-of-magnitude qualitative models, Hesitant Fuzzy Linguistic Term Sets (HFLTSs) were introduced in (Rodríguez, Martínez, and Herrera 2012) and a lattice structure is provided to the set of HFLTSs in (Montserrat-Adell et al.).

In this paper, inspired by previous woks over absolute order-of-magnitude qualitative models (Agell et al. 2012; Prats et al. 2014), we present an extension of the set of HFLTSs, $\overline{\mathcal{H}_\mathcal{S}}$, based on an equivalence relation on the usual set of HFLTSs. This enables us to establish differences between non-compatible HFLTSs. An order relation and two closed operation over this set are also introduced to define a new lattice structure in $\overline{\mathcal{H}_\mathcal{S}}$. A distance between HFLTSs is defined based on the lattice of $\overline{\mathcal{H}_\mathcal{S}}$.

This structures may be very useful in management situations such as marketing or human resources problems, where order-of-magnitude labels are used to assess. For instance, a common linguistic scale in the human resources field is: outstanding, exceeds expectations, meets expectations, below expectations and unsatisfactory.

The rest of this paper is organized as follows: first, Section 1 presents a brief review of HFLTSs and its lattice structure. The lattice of the extended set of HFLTSs is introduced in Section 2. In Section 3, the distances between HFLTSs are defined. Lastly, Section 4 contains the main conclusions and lines of future research.

1 The Lattice of Hesitant Fuzzy Linguistic Term Sets

In this section we present a brief review of some concepts about HFLTSs already presented in the literature that are used throughout this paper (Montserrat-Adell et al. ; Rodríguez, Martínez, and Herrera 2012).

From here on, let $\mathcal{S}$ denote a finite total ordered set of linguistic terms, $\mathcal{S} = \{a_1, \dots, a_n\}$ with $a_1 < \dots < a_n$.

Definition 1. (Rodríguez, Martínez, and Herrera 2012) A *hesitant fuzzy linguistic term set (HFLTS)* over $\mathcal{S}$ is a subset of consecutive linguistic terms of S, i.e. $\{x \in \mathcal{S} \mid a_i \leq x \leq a_j\}$, for some $i, j \in \{1, \dots, n\}$ with $i \leq j$.

*Corresponding author

The HFLTS $\mathcal{S}$ is called the *full HFLTS*. Moreover, the empty set $\{\} = \emptyset$ is also considered as a HFLTS and it is called the *empty HFLTS.*

For the rest of this paper, the non-empty HFLTS, $H = \{x \in \mathcal{S} \mid a_i \leq x \leq a_j\}$, is denoted by $[a_i, a_j]$. Note that, if $j = i$, the HFLTS $[a_i, a_i]$ is expressed as the singleton $\{a_i\}$.

The set of all the possible HFLTSs over $\mathcal{S}$ is denoted by $\mathcal{H}_{\mathcal{S}}$, being $\mathcal{H}^*_{\mathcal{S}} = \mathcal{H}_{\mathcal{S}} - \{\emptyset\}$ the set of all the non-empty HFLTSs. This set is provided with a lattice structure in (Montserrat-Adell et al.) with the two following operations: on the one hand, the *connected union of two HFLTSs,* $\sqcup$, which is defined as the least element of $\mathcal{H}_{\mathcal{S}}$, based on the subset inclusion relation $\subseteq$, that contains both HFLTSs, and on the other hand, the intersection of HFLTSs, $\cap$, which is defined as the usual intersection of sets. Notice that the usual union of sets cannot be considered given that it may not result a HFLTS. In addition, the reason of including the empty HFLTS in $\mathcal{H}_{\mathcal{S}}$ is to make the intersection of HFLTSs a closed operation in $\mathcal{H}_{\mathcal{S}}$.

For the sake of comprehensiveness, let us introduce the following example that is used throughout all this paper to depict all the concepts defined.

Example 1. Given the common set of linguistic labels, used in performance appraisal processes for human resources, $\mathcal{S} = \{a_1, a_2, a_3, a_4, a_5\}$, being a_1 = *unsatisfactory*, a_2 = *below expectations*, a_3 = *meets expectations*, a_4 = *exceeds expectations*, a_5 = *outstanding*, possible linguistic assessments and their corresponding HFLTSs by means of $\mathcal{S}$ would be:

Assessments	*HFLTSs*
A = *"below or meets expectations"*	$H_A = [a_2, a_3]$
B = *"below expectations"*	$H_B = \{a_2\}$
C = *"above meets expectations"*	$H_C = [a_4, a_5]$
D = *"below meets expectations"*	$H_D = [a_1, a_2]$
E = *"not outstanding"*	$H_E = [a_1, a_4]$

2 The Extended Lattice of Hesitant Fuzzy Linguistic Term Sets

With the aim of describing differences between couples of HFLTSs with empty intersections, an extension of the intersection of HFLTSs is presented in this section, resulting their intersection if it is not empty or a new element that we will call *negative HFLTS* related to the rift, or gap, between them if their intersection is empty. In order to present said extension of the intersection between HFLTSs, we first need to introduce the mathematical structure that allows us to define it as a closed operation. To this end, we define the extended set of HFLTSs in an analogous way to how integer numbers are defined based on an equivalence relation on the natural numbers. To do so, we first present some needed concepts:

Definition 2. Given two non-empty HFLTSs, $H_1, H_2 \in \mathcal{H}^*_{\mathcal{S}}$, we define:

(a) The *gap between H_1 and H_2* as:

$$gap(H_1, H_2) = (H_1 \sqcup H_2) \cap \overline{H_1} \cap \overline{H_2},$$

where $\overline{H}$ represents the complement of H.

(b) H_1 and H_2 are *consecutive* if and only if $H_1 \cap H_2 = \emptyset$ and $gap(H_1, H_2) = \emptyset$.

Proposition 1. Given two non-empty HFLTSs, $H_1, H_2 \in \mathcal{H}^*_{\mathcal{S}}$, the following properties are met:

1. $gap(H_1, H_2) = gap(H_2, H_1)$.
2. If $H_1 \subseteq H_2$, $gap(H_1, H_2) = \emptyset$.
3. If $H_1 \cap H_2 \neq \emptyset$, $gap(H_1, H_2) = \emptyset$.
4. If $H_1 \cap H_2 = \emptyset$, $gap(H_1, H_2) \neq \emptyset$ or H_1 and H_2 are consecutive.
5. If H_1 and H_2 are consecutive, there exist $j \in \{2, \ldots, n-1\}$, $i \in \{1, \ldots, j\}$ and $k \in \{j+1, \ldots, n\}$, such that $H_1 = [a_i, a_j]$ and $H_2 = [a_{j+1}, a_k]$ or $H_2 = [a_i, a_j]$ and $H_2 = [a_{j+1}, a_k]$.

Proof. The proof is straightforward. □

Note that neither $[a_1, a_j]$ nor $[a_i, a_n]$ can ever be the result of the *gap* between two HFLTSs for any i and for any j.

Notation. Given two consecutive HFLTSs, $H_1 = [a_i, a_j]$ and $H_2 = [a_{j+1}, a_k]$, then $\{a_j\}$ and $\{a_{j+1}\}$ are named as the linguistic terms that provide the consecutiveness of H_1, H_2.

Example 2. Following Example 1, $gap(H_B, H_C) = \{a_3\}$, while the HFLTSs H_A and H_C are consecutive and their consecutiveness is given by $\{a_3\}$ and $\{a_4\}$.

Definition 3. Given two pairs of non-empty HFLTSs, (H_1, H_2) and (H_3, H_4), the *equivalence relation* $\sim$*,* is defined as:

$$(H_1, H_2) \sim (H_3, H_4)$$
$$\Updownarrow$$
$$\begin{cases} H_1 \cap H_2 = H_3 \cap H_4 \neq \emptyset \\ \vee \\ gap(H_1, H_2) = gap(H_3, H_4) \neq \emptyset \\ \vee \\ \text{both pairs are consecutive and} \\ \text{their consecutiveness is provided} \\ \text{by the same linguistic terms} \end{cases}$$

It can be easily seen that $\sim$ relates couples of non-empty HFLTSs with the same intersection if they are compatible, with consecutiveness provided by the same linguistic terms if they are consecutive and with the same *gap* between them in the case that they are neither compatible nor consecutive.

Example 3. Following Example 1, the pairs of HFLTSs (H_A, H_B) and (H_A, H_D) are related according to $\sim$ given that they have the same intersection, $\{a_2\}$. Additionally, $(H_C, H_B) \sim (H_C, H_D)$ since they have the same *gap* between them, $\{a_3\}$.

Applying this equivalence relation over the set of all the pairs of non-empty HFLTSs, we get the quotient set $(\mathcal{H}^*_{\mathcal{S}})^2/\sim$, whose equivalence classes can be labeled as:

- $[a_i, a_j]$ for the class of all pairs of compatible non-empty HFLTSs with intersection $[a_i, a_j]$, for all $i, j = 1, \ldots, n$ with $i \le j$.
- $-[a_i, a_j]$ for the class of all pairs of incompatible non-empty HFLTSs whose *gap* is $[a_i, a_j]$, for all $i, j = 2, \ldots, n-1$ with $i \le j$.
- α_i for the class of all pairs of consecutive non-empty HFLTSs whose consecutiveness is provided by $\{a_i\}$ and $\{a_{i+1}\}$, for all $i = 1, \ldots, n-1$.

For completeness and symmetry reasons, $(\mathcal{H}_{\mathcal{S}}^*)^2/\sim$ is represented as shown in Figure 1 and stated in the next definition.

Example 4. Subsequent to this labeling, and following Example 1, the pair (H_C, H_B) belongs to the class $-\{a_3\}$ and so does the pair (H_C, H_D). The pair (H_C, H_A) belongs to the class α_3 and the pair (H_C, H_E) belongs to the class $\{a_4\}$.

Definition 4. Given a set of ordered linguistic term sets $\mathcal{S} = \{a_1, \ldots, a_n\}$, the *extended set of HFLTSs, $\overline{\mathcal{H}_{\mathcal{S}}}$,* is defined as:

$$\overline{\mathcal{H}_{\mathcal{S}}} = (-\mathcal{H}_{\mathcal{S}}^*) \cup \mathcal{A} \cup \mathcal{H}_{\mathcal{S}}^*,$$

where $-\mathcal{H}_{\mathcal{S}}^* = \{-H \mid H \in \mathcal{H}_{\mathcal{S}}^*\}$ and $\mathcal{A} = \{\alpha_0, \ldots, \alpha_n\}$.

In addition, by analogy with real numbers $-\mathcal{H}_{\mathcal{S}}^*$ is called the *set of negative HFLTSs,* $\mathcal{A}$ is called the *set of zero HFLTSs,* and, from now on, $\mathcal{H}_{\mathcal{S}}^*$ is called the *set positive HFLTSs.*

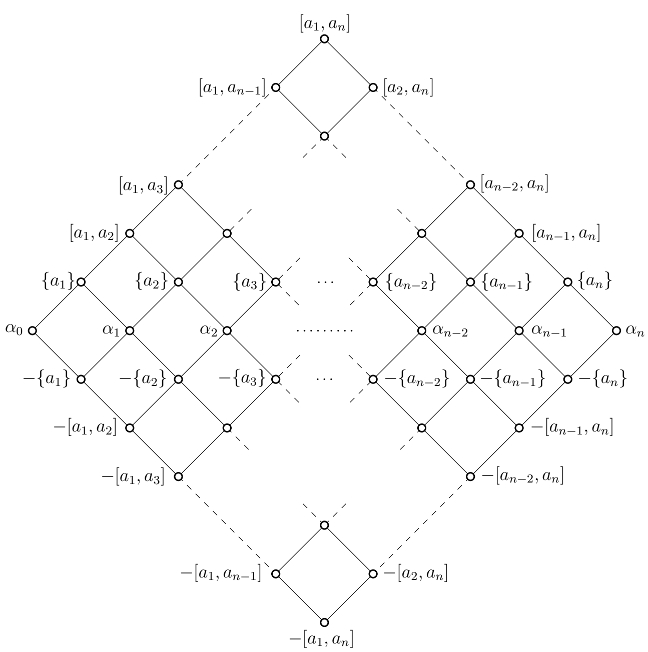

Figure 1: Graph of the extended set of HFLTSs.

Note that HFLTSs can be characterized by couples of zero HFLTSs. This leads us to introduce a new notation for HFLTSs:

Notation. Given a HFLTS, $H \in \overline{\mathcal{H}_{\mathcal{S}}}$, it can be expressed as $H = \langle \alpha_i, \alpha_j \rangle$, where the first zero HFLTS identifies the bottom left to top right diagonal and the second one identifies the top left to bottom right diagonal. Thus, $\langle \alpha_i, \alpha_j \rangle$ corresponds with $[a_{i+1}, a_j]$ if $i < j$, with $-[a_{i+1}, a_j]$ if $i > j$ and α_i if $i = j$.

This notation is used in the following definition that we present in order to latter introduce an order relation within $\overline{\mathcal{H}_{\mathcal{S}}}$.

Definition 5. Given $H \in \overline{\mathcal{H}_{\mathcal{S}}}$ described by $\langle \alpha_i, \alpha_j \rangle$ the *coverage of H* is defined as:

$$cov(H) = \{\langle \alpha_{i'}, \alpha_{j'} \rangle \in \overline{\mathcal{H}_{\mathcal{S}}} \mid i' \ge i \wedge j' \le j\}.$$

Example 5. The coverage of H_A from Example 1 can be seen in Figure 2.

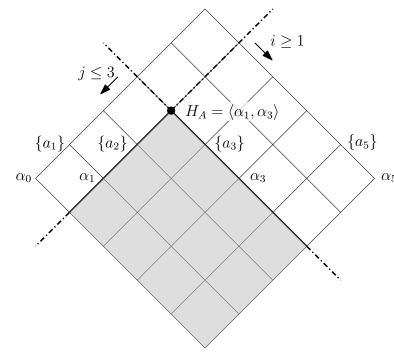

Figure 2: Coverage of H_A.

The concept of coverage of a HFLTS enables us to define the *extended inclusion relation* between elements of $\overline{\mathcal{H}_{\mathcal{S}}}$.

Definition 6. The *extended inclusion relation in $\overline{\mathcal{H}_{\mathcal{S}}}$, $\preccurlyeq$,* is defined as:

$$\forall H_1, H_2 \in \overline{\mathcal{H}_{\mathcal{S}}}, \quad H_1 \preccurlyeq H_2 \iff H_1 \in cov(H_2).$$

Note that, restricting to only the positive HFLTSs, the extended inclusion relation coincides with the usual subset inclusion relation. According to this relation in $\overline{\mathcal{H}_{\mathcal{S}}}$, we can define the *extended connected union* and the *extended intersection* as closed operations within the set $\overline{\mathcal{H}_{\mathcal{S}}}$ as follows:

Definition 7. Given $H_1, H_2 \in \overline{\mathcal{H}_{\mathcal{S}}}$, the *extended connected union of H_1 and H_2, $H_1 \sqcup H_2$,* is defined as the least element that contains H_1 and H_2, according to the extended inclusion relation.

Definition 8. Given $H_1, H_2 \in \overline{\mathcal{H}_{\mathcal{S}}}$, the *extended intersection of H_1 and H_2, $H_1 \sqcap H_2$,* is defined as the largest element being contained in H_1 and H_2, according to the extended inclusion relation.

It is straightforward to see that the extended connected union of two positive HFLTSs coincides with the connected union presented in (Montserrat-Adell et al.). This justifies the use of the same symbol. About the extended intersection of two positive HFLTSs, it results the usual intersection of sets if they overlap and the *gap* between them if they do not overlap. Notice that the empty HFLTS is not needed to make the extended intersection a closed operation in $\overline{\mathcal{H}_{\mathcal{S}}}$.

Proposition 2. Given two non-empty HFLTSs, $H_1, H_2 \in \mathcal{H}_{\mathcal{S}}^*$, if $H_1 \preccurlyeq H_2$, then $H_1 \sqcup H_2 = H_2$ and $H_1 \sqcap H_2 = H_1$.

Proof. The proof is straightforward. □

Example 6. Figure 3 provides an example with the extended connected union and the extended intersection of H_B and H_C and of H_A and H_E from Example 1: $H_B \sqcup H_C = [a_2, a_5]$, $H_B \sqcap H_C = -\{a_3\}$, $H_A \sqcup H_E = H_E$ and $H_A \sqcap H_E = H_A$. Note that $H_B \cup H_C = \{a_2, a_4, a_5\}$ is not a HFLTS.

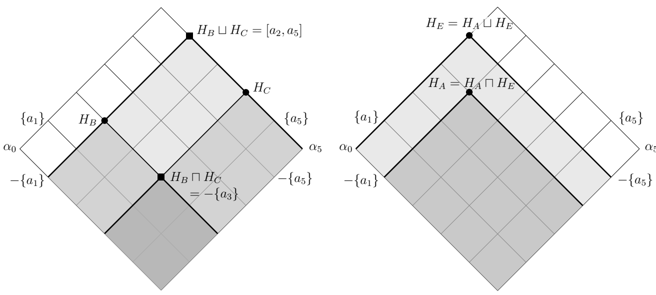

Figure 3: $\sqcup$ and $\sqcap$ of HFLTSs.

Proposition 3. $(\overline{\mathcal{H}_{\mathcal{S}}}, \sqcup, \sqcap)$ is a distributive lattice.

Proof. According to their respective definitions, both operations, $\sqcup$ and $\sqcap$, are trivially commutative and idempotent.

The associative property of $\sqcup$ is met since $(H_1 \sqcup H_2) \sqcup H_3 = H_1 \sqcup (H_2 \sqcup H_3)$ given that both parts equal the least element that contains H_1, H_2 and H_3. About the associativeness of $\sqcap$, $(H_1 \sqcap H_2) \sqcap H_3 = H_1 \sqcap (H_2 \sqcap H_3)$ given that in both cases it results the largest element contained in H_1, H_2 and H_3.

Finally, the absorption laws are satisfied given that: on the one hand $H_1 \sqcup (H_1 \sqcap H_2) = H_1$ given that $H_1 \sqcap H_2 \preccurlyeq H_1$ and on the other hand $H_1 \sqcap (H_1 \sqcup H_2) = H_1$ given that $H_1 \preccurlyeq H_1 \sqcup H_2$.

Furthermore, the lattice $(\overline{\mathcal{H}_{\mathcal{S}}}, \sqcup, \sqcap)$ is distributive given that none of its sublattices are isomorphic to the diamond lattice, M_3, or the pentagon lattice, N_5. □

3 A Distance between Hesitant Fuzzy Linguistic Term Sets

In order to define a distance between HFLTSs, we introduce a generalization of the concept of cardinal of a positive HFLTS to all the elements of the extended set of HFLTSs.

Definition 9. Given $H \in \overline{\mathcal{H}_{\mathcal{S}}}$, the *width of* H is defined as:

$$\mathcal{W}(H) = \begin{cases} card(H) & if \quad H \in \mathcal{H}_{\mathcal{S}}^*, \\ 0 & if \quad H \in \mathcal{A}, \\ -card(-H) & if \quad H \in (-\mathcal{H}_{\mathcal{S}}^*). \end{cases}$$

Note that the width of a HFLTS could be related as well with the height on the graph of $\overline{\mathcal{H}_{\mathcal{S}}}$, associating the zero HFLTSs with height 0, the positive HFLTSs with positive heights and the negative HFLTSs with negative values of heights as shown in Figure 4.

Proposition 4. $D(H_1, H_2) = \mathcal{W}(H_1 \sqcup H_2) - \mathcal{W}(H_1 \sqcap H_2)$ provides a distance in the lattice $(\overline{\mathcal{H}_{\mathcal{S}}}, \sqcup, \sqcap)$.

Proof. $D(H_1, H_2)$ defines a distance given that it is equivalent to the geodesic distance in the graph $\overline{\mathcal{H}_{\mathcal{S}}}$. The geodesic distance between H_1 and H_2 is the length of the shortest path to go from H_1 to H_2. Due to the fact that $H_1 \sqcap H_2 \preccurlyeq H_1 \sqcup H_2$, $\mathcal{W}(H_1 \sqcup H_2) - \mathcal{W}(H_1 \sqcap H_2)$ is the length of the minimum path between $H_1 \sqcup H_2$ and $H_1 \sqcap H_2$. Thus, we have to check that the length of the shortest path between $H_1 \sqcup H_2$ and $H_1 \sqcap H_2$ coincides with the length of the shortest path between H_1 and H_2.

If one of them belong to the coverage of the other one, let us suppose that $H_1 \preccurlyeq H_2$, then $H_1 \sqcup H_2 = H_2$ and $H_1 \sqcap H_2 = H_1$ and the foregoing assertion becomes obvious. If not, H_1, $H_1 \sqcup H_2$, H_2 and $H_1 \sqcap H_2$ define a parallelogram on the graph. Two consecutive sides of this parallelogram define the shortest path between $H_1 \sqcup H_2$ and $H_1 \sqcap H_2$ while two other consecutive sides of the same parallelogram define the shortest path between H_1 and H_2. Thus, the assertion becomes true as well. □

Proposition 5. Given two HFLTSs, $H_1, H_2 \in \overline{\mathcal{H}_{\mathcal{S}}}$, then $0 \leq D(H_1, H_2) \leq 2n$. If, in addition, $H_1, H_2 \in \mathcal{H}_{\mathcal{S}}^*$, then $0 \leq D(H_1, H_2) \leq 2n - 2$.

Proof. For the lower bound, notice that since $H_1 \sqcap H_2 \subseteq H_1 \sqcup H_2$, then $\mathcal{W}(H_1 \sqcap H_2) \leq \mathcal{W}(H_1 \sqcup H_2)$, and therefore $D(H_1, H_2) \geq 0$.

For the upper bound, if $H_1, H_2 \in \overline{\mathcal{H}_{\mathcal{S}}}$, then, the most distant pair is α_0 and α_n. Then,

$$\mathcal{W}(\alpha_0 \sqcup \alpha_n) - \mathcal{W}(\alpha_0 \sqcap \alpha_n) =$$

$$\mathcal{W}([a_1, a_n]) - \mathcal{W}(-[a_1, a_n]) =$$

$$n - (-n) = 2n.$$

If $H_1, H_2 \in \mathcal{H}_{\mathcal{S}}^*$, then, the most distant pair is $\{a_1\}$ and $\{a_n\}$. Then,

$$\mathcal{W}(\{a_1\} \sqcup \{a_n\}) - \mathcal{W}(\{a_1\} \sqcap \{a_n\}) =$$

$$\mathcal{W}([a_1, a_n]) - \mathcal{W}(-[a_2, a_{n-1}]) =$$

$$n - (-(n-2)) = 2n - 2.$$

□

Notice that for positive HFLTSs, $D(H_1, H_2)$ coincides with the distance $D_2(H_1, H_2)$ introduced in (Montserrat-Adell et al.). Additionally, in this case, the distance presented can also be calculated as $D([a_i, a_j], [a_{i'}, a_{j'}]) = |i - i'| + |j - j'|$.

Example 7. Figure 4 shows the width of the extended connected union and the extended intersection of H_B and H_C from Example 1. According to these results, $D(H_B, H_C) = \mathcal{W}(H_B \sqcup H_C) - \mathcal{W}(H_B \sqcap H_C) = 4 - (-1) = 5$.

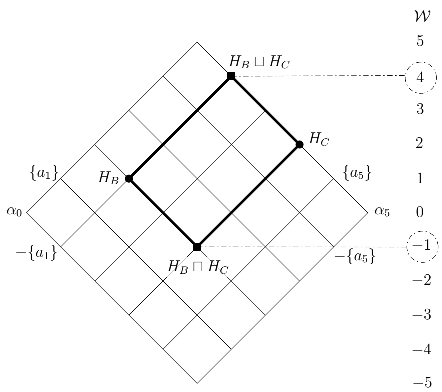

Figure 4: Distance between HFLTSs.

4 Conclusions and future research

This paper presents, inspired by previous works over absolute order-of-magnitude qualitative models, an extension of the set of Hesitant Fuzzy Linguistic Term Sets by introducing the concepts of negative and zero HFLTSs to capture differences between pairs of non-compatible HFLTSs. This extension enables the introduction of a new operation studying the intersection and the gap between HFLTSs at the same time. This operation is used to define a distance between HFLTSs that allows us to analyze differences between the assessments given by a group of decision makers.

There is, nowadays, a wide range of areas of application for distances between linguistic assessments, from managerial to medical or engineering. Future research is focused in two main directions. First, the study of the consensus level of the total group assessments to analyze the agreement or disagreement within the group. And secondly, a real case study will be performed in the marketing research area to examine consensus and heterogeneities in consumers' preferences.

References

Agell, N.; Sánchez, M.; Prats, F.; and Roselló, L. 2012. Ranking multi-attribute alternatives on the basis of linguistic labels in group decisions. *Information Sciences* 209:49–60.

Deschrijver, G., and Kerres, E. E. 2003. On the relationship between some extensions of fuzzy set theory. *Fuzzy Sets and Systems* 133:227–235.

Espinilla, M.; Liu, J.; and Martínez, L. 2011. An extended hierarchical linguistic model for decision-making problems. *Computational Intelligence* 27(3):489–512.

Forbus, K. D. 1996. *Qualitative Reasoning*. CRC Handbook of Computer Science and Engineering, CRC Press,.

Greenfield, S., and Chiclana, F. 2013. Accuracy and complexity evaluation of defuzzification strategies for the discretised interval type-2 fuzzy set. *International Journal of Approximate Reasoning* 54(8):1013–1033.

Herrera, F.; Herrera-Viedma, E.; and Martínez, L. 2008. A fuzzy linguistic methodology to deal with unbalanced linguistic term sets. *IEEE Transactions on Fuzzy Systems* 16(2):354–370.

Herrera-Viedma, E.; Herrera, F.; and Chiclana, F. 2002. A consensus model for multiperson decision making with different preference structures. *IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans* 32:394–402.

Montserrat-Adell, J.; Agell, N.; Sánchez, M.; Prats, F.; and Ruiz, F. J. Modeling group assessments by means of hesitant fuzzy linguistic term sets. (To appear in Journal of Applied Logic).

Parreiras, R. O.; Ekel, P. Y.; Martini, J. S. C.; and Palhares, R. M. 2010. A flexible consensus scheme for multicriteria group decision making under linguistic assessments. *Information Sciences* 180(7):1075–1089.

Prats, F.; Roselló, L.; Sánchez, M.; and Agell, N. 2014. Using l-fuzzy sets to introduce information theory into qualitative reasoning. *Fuzzy Sets and Systems* 236(1):73–90.

Rodríguez, R. M.; Martínez, L.; and Herrera, F. 2012. Hesitant fuzzy linguistic terms sets for decision making. *IEEE Transactions on Fuzzy Systems* 20(1):109–119.

Tang, Y., and Zheng, J. 2006. Linguistic modelling based on semantic similarity relation among linguistic labels. *Fuzzy Sets and Systems* 157(12):1662–1673.

Travé-Massuyès, L., and Dague, P. 2003. *Modèles et raisonnements qualitatifs*. Paris, France: Ed. Hermès.

Travé-Massuyès, L.; Prats, F.; Sanchez, M.; and Agell, N. 2005. Relative and absolute order-of-magnitude models unified. *Annals of mathematics and artificial intelligence* 45(1):323–341.