DECOMPOSITION OF SEPARABLE CONCAVE STRUCTURING FUNCTIONS

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Abstract. This paper presents a decomposition scheme for a large class of greyscale structuring elements from mathematical morphology. In contrast with many existing decomposition schemes, our method is valid in the continuous domain. Conditions are given under which this continuous method can be properly discretized. The class of functions that can be decomposed with our method contains the class of quadratic functions, that are of major importance in, for instance, distance transforms and morphological scale space. In the continuous domain, the size of the structuring elements resulting from the decomposition, can be chosen arbitrarily small. For functions from the mentioned class, that can be separated along the standard image axes, a discrete decomposition in 3×3 elements can be guaranteed.

Key words: Mathematical Morphology, Structuring Element Decomposition, Concave Structuring Elements.

1. Introduction

In this paper we study the decomposition of a large class of structuring functions often used in morphological image processing. We study decomposition of continuous functions and prove that a subset of the class of concave functions can be decomposed into a dilation sequence of functions with finite effective domain. Only then we show what are the requirements for proper discretization of the proposed decomposition scheme.

In mathematical morphology [18, 7], concave structuring functions play an important role. Matheron [14] already showed that convexity of structuring sets is needed to axiomatize the concept of size. This analysis lead to the notion of granulometries. Concave functions are needed to extend the notion of granulometries to grey level functions [11]. In [10], Xu presents a decomposition scheme for *convex* polygon-shaped structuring elements in binary morphology.

Concave structuring functions play an important role in the calculation of (Euclidean) distance transforms [2]. Sternberg [19] showed that the Euclidean distance transform can be calculated by dilating the indicator function of a set with a cone shaped structuring function. The cone function $c(x, y) = -\sqrt{x^2 + y^2}$ encodes the distance to its center. The infinite support function c can be decomposed into the sequence $c = t \oplus t \oplus \cdots$ where t is the "top" of the cone $t(x, y) = c(x, y)$ for $x^2 + y^2 \le 1$ and $t(x,y) = -\infty$ elsewhere. Unfortunately this decomposition cannot be properly discretized, i.e. discretizing the tops and then dilating leads to a different result then first dilating and then discretizing. Therefore a *chamfer distance transform* [1, 26] can only be an approximation of the Euclidean distance transform.

Dilating the indicator function of a set with the square of the cone function leads to the square of the distance transform [22] of that set. Huang[8] showed that the discretized *squared* cone *can* be decomposed into a sequence of finite support discrete structuring functions. In this paper we will give a geometrical *continuous* construction that is a generalization of this result in the sense that our result can be used for a larger class of (continuous) concave functions (not only the parabola).

The parabolic function is not only important because it can be used to calculate the distance transform. It has been shown by van den Boomgaard [24] and Jackway [9] that the parabola in a very specific sense is the morphological analogue of the Gaussian function as used in linear convolutions [22, 24].

Decomposition of structuring elements ([28]) has a lot of literature devoted to it. Some of the reported results deal with continuous structuring elements. For example the decomposition of convex symmetric polygons into the dilation of the edge line segments [18] is simple to prove for polygons in $\mathbb{R}^2$. Also the logarithmic decomposition of a convex set into the dilation of the set with its extreme elements (the vertices of a polygon) [15, 25] is proved for continuous sets (and the conditions for proper discretization are given in [21]). Most of the results on decomposition, however, are focused on the decomposition of discrete sets [5, 20, 12, 16, 6]. Whereas the decomposition of continuous sets tend to be of a geometrical nature, the proofs for the decomposition of discrete sets tend to be of an algebraic nature [17].

Decomposition into small structuring elements is important from a practical point of view. Even for the Euclidean distance transform, that can be implemented quite efficiently using the dimensional decomposition (see [23]), the decomposition into small 3×3 elements is profitable as it allows for inhomogeneous distance transforms [26]. These are for instance needed in the watershed algorithm while keeping track of the distance traveled from the starting marker points [27].

2. Decomposition

Consider the dilation of a function f with respect to the structuring function g:

$$(f \oplus g)(x) = \bigvee_{y \in \mathbb{R}^n} f(x-y) + g(y).$$

The effective domain of the structuring function g is the set of points y where $g(y) \neq -\infty$. Note that only points in the effective domain of g need to be considered in calculating the dilation result.

The structuring functions considered in this paper are all concave. Concave structuring functions generalize the notion of convex sets to the domain of grey value images. A function $g : \mathbb{R}^n \to \mathbb{R}$ is concave if $g(tx + (1-t)y) \geq tg(x) + (1-t)g(y)$ for $x, y \in \mathbb{R}^n$ and $0 \leq t \leq 1$. A concave function is *proper concave* if $g(x) > -\infty$ for at least one x and $g(x) < \infty$ for all x.

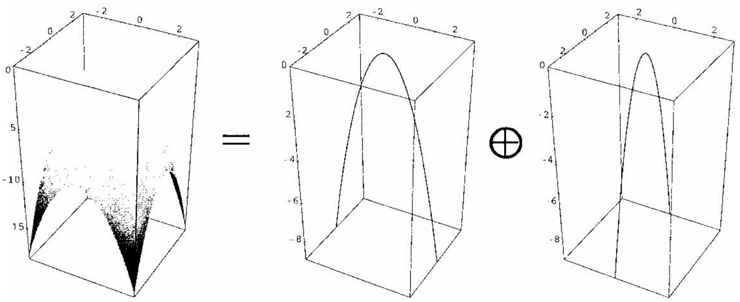

Fig. 1. Example of a separation of a parabola g into $\chi_{\vec{v}} g_{\vec{v}}$ and $\chi_{\vec{w}} g_{\vec{w}}$

We restrict ourselves to structuring functions g that can be separated in one-dimensional proper concave functions $g_{\vec{v}}$ and $g_{\vec{w}}$. The separation process makes use of the embedding operator $\chi_{\vec{v}}$, to embed one-dimensional functions into two dimensional space (see [23]):

Definition 1 *Let l be a one dimensional real function. The operator $\chi_{\vec{v}}$ (with $\vec{v}$ a direction vector, $\|\vec{v}\| = 1$) embeds the function l into 2-dimensional space, resulting in the function $\chi_{\vec{v}} l : \mathbb{R}^2 \to \mathbb{R}$:*

$$\chi_{\vec{v}} l(\vec{x}) = \begin{cases} l(\|\vec{x}\|) & , \textit{if } \vec{x} \textit{ parallel to } \vec{v} \\ -\infty & , \textit{otherwise} \end{cases} \tag{1}$$

The structuring function g is called separable if it can be separated in functions $g_{\vec{v}}$ and $g_{\vec{w}}$ such that:

$$g = \chi_{\vec{v}} g_{\vec{v}} \oplus \chi_{\vec{w}} g_{\vec{w}}, \tag{2}$$

(see figure 1 for an example)

Notice that when we take two arbitrary proper concave functions $g_{\vec{v}}$ and $g_{\vec{w}}$ the resulting function $g(\vec{x}) = \chi_{\vec{v}} g_{\vec{v}} \oplus \chi_{\vec{w}} g_{\vec{w}}$ is always a proper concave function.

The vectors $\vec{v}$ and $\vec{w}$ in the separation must be linearly independent. Functions $f : \mathbb{R}^2 \to \mathbb{R}$ which can be written as $f(x, y) = f_1(x) + f_2(y)$, with $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ are a subclass of the functions that can be separated cf. equation 2.

2.1. The decomposition scheme

In theorem 1 we will prove that a one dimensional proper concave function g can be decomposed in two concave functions u and r, such that $f = u \oplus r$ (under certain conditions). This function u always has a finite effective domain. Following this theorem allows for the decomposition of $g_{\vec{v}}$ and $g_{\vec{w}}$:

$$g_{\vec{v}} = (u_{\vec{v}} \oplus r_{\vec{v}}), \qquad g_{\vec{w}} = (u_{\vec{w}} \oplus r_{\vec{w}}).$$

Now dilation of image I with function g is $I \oplus g = I \oplus (\chi_{\vec{v}} g_{\vec{v}} \oplus \chi_{\vec{w}} g_{\vec{w}})$, with $\chi_{\vec{v}} g_{\vec{v}} \oplus \chi_{\vec{w}} g_{\vec{w}} = \chi_{\vec{v}}(u_{\vec{v}} \oplus r_{\vec{v}}) \oplus \chi_{\vec{w}}(u_{\vec{w}} \oplus r_{\vec{w}})$. The embedding operator can be distributed over the dilation, so $g = (\chi_{\vec{v}} u_{\vec{v}} \oplus \chi_{\vec{v}} r_{\vec{v}}) \oplus (\chi_{\vec{w}} u_{\vec{w}} \oplus \chi_{\vec{w}} r_{\vec{w}})$, and since

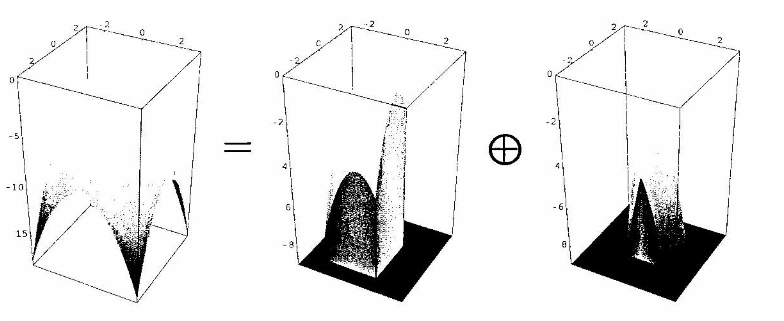

Fig. 2. Example of a decomposition of a parabola g into u_1 and r_1

dilation is commutative and associative $g = \chi_{\vec{v}} u_{\vec{v}} \oplus \chi_{\vec{w}} u_{\vec{w}} \oplus \chi_{\vec{v}} r_{\vec{v}} \oplus \chi_{\vec{w}} r_{\vec{w}}$. If we define $u_1 = \chi_{\vec{v}} u_{\vec{v}} \oplus \chi_{\vec{w}} u_{\vec{w}}$ and $r_1 = \chi_{\vec{v}} r_{\vec{v}} \oplus \chi_{\vec{w}} r_{\vec{w}}$, then $I \oplus g = (I \oplus u_1) \oplus r_1$. (See for an example figure 2.) Since $r_{\vec{v}}$ and $r_{\vec{w}}$ are proper concave functions, the same process can be repeated and the dilation with g can be carried out as follows $I \oplus g = I \oplus u_1 \oplus u_2 \oplus r_2$. For n decomposition steps this results in

$$I \oplus g = I \oplus u_1 \oplus \ldots \oplus u_n \oplus r_n. \tag{3}$$

Only in case the function g has a finite effective domain, it can be decomposed in a finite number of functions $u_1 \ldots u_n$, such that r_n is the pulse function. For functions with an infinite effective domain, the rest function r_n always has an infinite effective domain.

2.2. DECOMPOSITION OF ONE DIMENSIONAL CONCAVE STRUCTURING FUNCTIONS

The main theorem of this section gives a decomposition of one dimensional proper concave functions f into two proper concave functions u and r, such that $f = u \oplus r$. In this decomposition u always has a finite effective domain, while r only has a finite effective domain if f has a finite effective domain. To simplify the decomposition, we assume that $f(x) < 0$, except for $x = 0$, where $f(0) = 0$. For proper concave functions this comes down to translating the function such that the maximum is obtained in the origin. Since the decomposed function is used for dilation, this only results in a simple translation of the result.

The function u is constructed from f as follows

$$u(x) = \begin{cases} f(x) & \text{, if } x_1^* \le x \le x_2^* \\ -\infty & \text{, otherwise} \end{cases} , \tag{4}$$

where x_1^* and x_2^* are chosen such that $f(x_1^*) = t_1$ and $f(x_2^*) = t_2$ for some real numbers $t_1 \le 0$, $t_2 \le 0$ and $x_1^* \le 0$, $x_2^* \ge 0$, see figure 3. The function r is constructed from f as follows

$$r(x) = \begin{cases} f(x + x_1^*) - t_1 & \text{, if } x < 0 \\ f(x + x_2^*) - t_2 & \text{, otherwise.} \end{cases} \tag{5}$$

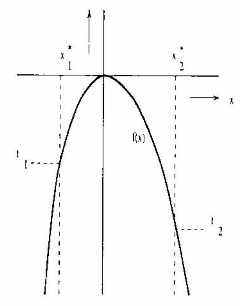

Fig. 3. Visualization of the symbols used in the decomposition.

The terms $-t_1$ and $-t_2$ translate the maximum of r to the origin so that the dilation $u \oplus r$ will not be a translated version of f. r is again a proper concave function, so the same decomposition scheme can be used to decompose r (with possible other values of x_1^* and x_2^*).

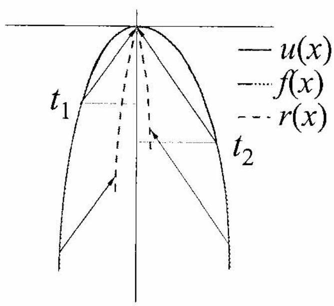

Fig. 4. Decomposition of f into u and r.

A non-trivial decomposition exists in case the effective domain of the concave function is of the form (a, b) where $a < 0$ and $b > 0$. Then t_1 and t_2 can always be found such that there exist $x_1^* \leq 0$ and $x_2^* \geq 0$, with $f(x_1^*) = t_1$ and $f(x_2^*) = t_2$.

Theorem 1 ensures that the decomposition given by equations 4 and 5 indeed results in a decomposition of f such that $f = u \oplus r$. The proof of this theorem, and all other proofs omitted in this paper, can be found in [4].

Theorem 1 *Let $f(x) : I\!R \to I\!R$ be a one dimensional proper concave function with $f(x) < 0$ except for $x = 0$, where $f(0) = 0$. Assume that there exist $x_1^* \leq 0$ and $x_2^* \geq 0$ such that $f(x_1^*) = t_1$ and $f(x_2^*) = t_2$ for some real numbers $t_1 \leq 0$, $t_2 \leq 0$. Now define*

$$u(x) = \begin{cases} f(x) & , \text{ if } x_1^* \leq x \leq x_2^* \\ -\infty & , \text{ otherwise} \end{cases}$$

and

$$r(x) = \begin{cases} f(x + x_1^*) - t & , \textit{if } x < 0 \\ f(x + x_2^*) - t & , \textit{otherwise.} \end{cases}$$

Then $u \oplus r(x) = f(x)$.

3. Discretization of the Decomposition Scheme

In this section we discuss the conditions under which the decomposition process can be properly discretized, i.e. the conditions under which dilating with the original discretized function is the same as dilation with the discretized results of the decomposition. First the definition of the discretization operator $\Delta_{\vec{c},\vec{d}}$. The vectors $\vec{c}$ and $\vec{d}$ generate the sampling points in the discretization grid.

Definition 2 *The discretization operator* $\Delta_{\vec{c},\vec{d}}$ *constructs a function* $\Delta_{\vec{c},\vec{d}}f : \mathbb{Z}^2 \to \mathbb{R}$ *from a function* $f : \mathbb{R}^2 \to \mathbb{R}$ *as follows:*

$$(\Delta_{\vec{c},\vec{d}}f)(p, q) = f(p\vec{c} + q\vec{d}).$$

Likewise, the discretization $\Delta_k f$ of a function $f : \mathbb{R} \to \mathbb{R}$ is constructed by $(\Delta_k f)(p) = f(pk)$.

Now the term "properly discretizable" of the decomposition $f = u \oplus r$ can be formalized as:

$$\Delta_{\vec{c},\vec{d}}f = \Delta_{\vec{c},\vec{d}}u \oplus \Delta_{\vec{c},\vec{d}}r.$$

In general the decomposition process can not be guaranteed to be properly discretizable.

The following lemma gives the conditions under which one dimensional decomposable functions can be properly discretized.

Lemma 1 *Let the decomposition for a function f be given by the parameters* x_1^* *and* x_2^* *and let the discretization be given by parameter d. Then*

$$\Delta_d f = \Delta_d u \oplus \Delta_d r$$

iff $\forall k \in \mathbb{Z}_0^-$, $kd \leq d\lceil \frac{x_1^*}{d} \rceil$:

$$f(kd) = f(kd - (d\lceil \frac{x_1^*}{d} \rceil - x_1^*)) + f(d\lceil \frac{x_1^*}{d} \rceil) - f(x_1^*)$$

and $\forall k \in \mathbb{Z}_0^+$, $kd \geq d\lfloor \frac{x_2^*}{d} \rfloor$:

$$f(kd) = f(kd + (x_2^* - d\lfloor \frac{x_2^*}{d} \rfloor)) + f(d\lfloor \frac{x_2^*}{d} \rfloor) - f(x_2^*).$$

For two-dimensional decomposable functions f that can be separated into $f_{\vec{v}}$ and $f_{\vec{w}}$, the question whether it can be discretized properly with parameters $\vec{v}$ and $\vec{w}$ boils down to the question whether $f_{\vec{v}}$ can be discretized with parameter $|\vec{v}|$ and $f_{\vec{w}}$ can be discretized with parameter $|\vec{w}|$.

If the discretization vectors $\vec{c}$ and $\vec{d}$ do not correspond with the separation parameters $\vec{v}$ and $\vec{w}$., it is still possible for the discretization of the decomposition scheme to be proper.

In the following theorem we use $[\vec{v}\vec{w}]$ to denote the 2×2 matrix whose column are the vectors $\vec{v}$ and $\vec{w}$.

Theorem 2 *Let f be a two-dimensional decomposable function that can be separated into* $f_{\vec{v}}$ *and* $f_{\vec{w}}$ *, for which* $\Delta_{|\vec{v}|}f_{\vec{v}} = \Delta_{|\vec{v}|}u_{\vec{v}} \oplus \Delta_{|\vec{v}|}r_{\vec{v}}$ *and* $\Delta_{|\vec{w}|}f_{\vec{w}} = \Delta_{|\vec{w}|}u_{\vec{w}} \oplus \Delta_{|\vec{w}|}r_{\vec{w}}$. *Then if* $\vec{v}$ *and* $\vec{w}$ *are two points from the lattice formed by* $\vec{c}$ *and* $\vec{d}$ *and there exists an integral matrix U such that det(U) = ±1 and* $[\vec{v}\vec{w}] = [\vec{c}\vec{d}]\,U$, *then*

$$\Delta_{\vec{c},\vec{d}}f = \Delta_{\vec{c},\vec{d}}u \oplus \Delta_{\vec{c},\vec{d}}r.$$

The well-known example decomposing a (discrete) parabola $g(p,q) = -(o^2 + q^2)$ that also follows directly from the theory presented in this paper is:

$$g = \left\{ \begin{array}{ccc} -2 & -1 & -2 \\ -1 & \underline{0} & -1 \\ -2 & -1 & -2 \end{array} \right\} \oplus \left\{ \begin{array}{ccc} -6 & -3 & -6 \\ -3 & \underline{0} & -3 \\ -6 & -3 & -6 \end{array} \right\} \oplus \cdots$$

Here we use the notation that within the curly brackets the values of a structuring function in the discrete sampling points are given. The origin is marked with an underscore. All the values that are not shown are implicitly assumed to be equal to $-\infty$.

The second example illustrates that non axes aligned quadratic structuring functions (QSF's) are also decomposable using our approach. Consider the QSF $g(p,q) = -p^2 + 2pq - 2q^2$. This QSF is decomposed as:

$$g = \left\{ \begin{array}{ccccc} & & -2 & -1 & -2 \\ & -1 & \underline{0} & -1 & \\ -2 & -1 & -2 & & \end{array} \right\} \oplus \left\{ \begin{array}{ccccc} & & -6 & -3 & -6 \\ & -3 & \underline{0} & -3 & \\ -6 & -3 & -6 & & \end{array} \right\} \oplus \cdots$$

4. Computational Complexity

In general, the dilation with a structuring function with effective domain nQ where Q is convex is of complexity $\mathcal{O}(M^2 n^2 |Q|)$, where $|Q|$ is the number of points in Q, and the dimension of the image is $M \times M$. If the structuring function nQ can be decomposed such that for the effective domain of nQ holds that

$$\underbrace{Q \oplus Q \oplus \ldots Q}_{n\ times},$$

the complexity is reduced to $\mathcal{O}(M^2 n |Q|)$.

Due to the nature of our separation process, the decomposition scheme always returns structuring functions with a parallelogram shaped effective domain. Since the size of the effective domains of the resulting functions of our decomposition scheme can be chosen at will, the domains can be chosen equal to Q, where the effective domain of the original structuring function is nQ. The decomposition scheme then reduces the complexity by a factor n.

5. Conclusions

In this paper we have presented a decomposition scheme for a large class of concave structuring functions. The results are valid in the continuous domain, but we have proved the requirements for proper discretization as well. The important class of quadratic structuring functions prove to be decomposable into a sequence of dilations with structuring functions restricted to a finite (small) effective domain. For the axis aligned parabola (an element of the class of quadratic functions) our proof is a generalization of the decomposition presented by Huang [8]. The main difference with existing approaches is that we have chosen a continuous geometrical view on decomposition instead of a discrete algebraic approach.

We have restricted our proofs to the functions that are separable by dimension. This is somewhat of a limitation that may well be eliminated using more elaborate proofs using the slope transform description of morphological operators [3] or equivalently (for concave functions) using the Fenchel conjugate functions (or the upper and lower slope transform) from convex analysis [13, 3]. Such an extension of the theory is left to future work.

We believe that the presented approach for decomposition of concave structuring functions provides an intuitive feeling for the decomposition (being a "cut-and-paste" procedure as illustrated in figure 4) that is fruitful for a deeper understanding of morphological operators modifying and probing the geometry of visual observations.

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