

Shape of an Arbitrary Finite Point Set in \mathbb{R}^2 *

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Abstract. We study the shape of a finite point set in \mathbb{R}^2 , where the points are not bound to a regular grid like \mathbb{Z}^2 . The shape of a connected point set in \mathbb{R}^2 is captured by its boundary. For a finite point set the boundary is a directed graph that connects points identified as boundary points. We argue that to serve as a proper boundary definition the directed graph should regulate scale, be minimal, have an increasing interior and be consistent with the boundary definition of connected objects.

We propose to use the directed variant of the α -shape as defined by Edelsbrunner et al (1983), which we call the α -graph. The α -graph is based on a generalization of the convex hull.

The computational aspects of the α -graph have been extensively studied, but little attention has been paid to the potential use of the α -graph as a shape descriptor or a boundary definition. In this paper we prove that the α -graph satisfies the aforementioned criteria. We also prove a relation between the α -graph and the opening scale space from mathematical morphology. In fact, the α -hull provides a generalization of this scale space.

Key words. shape analysis, boundary definition, computational geometry, mathematical morphology, scale space

1 Introduction

In this paper we consider the shape of a finite (nonconnected) point set S in \mathbb{R}^2 . The shape of a connected subset S' in \mathbb{R}^2 (called an *object*) is captured by its boundary. For a connected set this boundary has a unique definition. In fact, $s' \in S'$ is a point on the boundary $\partial S'$ whenever an arbitrary small disc centered at s' contains an element not in S' . In the connected case, all points that are not part of the boundary are part of the interior of S' . (By convention, the boundary is given a direction such that the interior lies to the left of the directed boundary $\partial S'$). As a consequence, the directed boundary defines a cyclic ordering of points on the boundary.

Using the boundary and interior definition as used for connected sets for the finite point set S would imply that the boundary of the point set S equals S . Thus the boundary would not yield an ordering of the elements. Furthermore, the

interior of the point set would be empty.

Now consider the situation where the points are bound to a regular grid (say, $S \subset \mathbb{Z}^2$). Then, it is common in image processing to define the boundary of S as the set of all points in S that have a neighbor $s \in \mathbb{Z}^2$ that is not in S . This definition is, however, not unique because one has to define the notion of neighbor. For points on a regular grid 4- or 8-connected neighborhoods are commonly used. The interior of the point set for this case is the set of points that have all their 4- or 8-connected neighbors in S .

However, when S is an arbitrary finite point set not bound to a regular grid but resulting from an irregular sampling of some connected object, neither of the preceding definitions can be used to define the boundary or interior of S .

The definition for connected sets and the definition for points on a regular grid have in common that they identify a subset of S as being boundary elements. From there a cyclic ordering of those boundary elements is defined. This suggests that the boundary of an arbitrary

*This work was supported by the Dutch Ministry of Economic Affairs through SPIN grants "3D Computer Vision" and "3D Image Analysis."

finite point set should be a directed graph containing as vertices those elements identified as boundary elements. The contour thus defined provides a local definition of the form of S and permits derivation of contour parameters such as length, orientation, and local curvature. We concentrate on a proper definition of the boundary of S .

A directed graph receiving considerable attention in the literature is the graph derived from the vertices and edges of the convex hull (denoted by $H(S)$) [13]. As a definition of the boundary of S , the convex hull has several limitations. Because $H(S)$ can assume only convex forms, essential shape information on concave regions is lost. On the other hand, the boundary of $H(S)$ may contain too much detail, prohibiting the reduction of the set to its essential form. Further, because the convex hull always consists of one closed face, it is unsuited for the description of a set of finite point clusters. These limitations are a consequence of the fact that the convex hull is unable to describe the boundary of the point set at different scales.

The point is illustrated by considering the boundary of a tree, when the tree is represented by a finite set of points. At some scale the boundary of the tree should be dictated by the individual leaves, where as at another scale only foliage should be identified. As a consequence, when no *a priori* information about the amount of detail is present, the boundary definition for a finite point set must include a scale parameter, which for reasons to become clear in the rest of the paper, will be denoted by α . Following the general scale space requirements [7], we will demand the boundary graph to be a one-parameter family with decreasing detail for decreasing scale (here in parameter α). Further, the boundary graph at a lower scale should be computable from a boundary graph at a higher scale. This is usually called *differential computation*.

It is important to note that this rationale behind the use of the scale parameter exists only for the type of finite point sets considered here. It is to be discriminated from those approaches to shape description of connected objects in \mathbb{R}^2 or points on a regular grid [1], [2], [7], [10]

in which a parameter of scale is used. Those methods are applied once the boundary of the object has been defined, which is not the case for the finite point sets considered here.

Thus far we found two important demands the boundary definition should satisfy: it should be a one-parameter family, and furthermore, as the convex-hull example showed, it should contain only the essential boundary information. Of course, as a consequence of the scale parameter used in the definition, the set of points giving essential boundary information is also a function of scale.

As of yet we have not considered the interior of the point set. No proper definition of the interior of an arbitrary finite point set exists in literature. We use the intuitive definition of the interior as being the region in \mathbb{R}^2 in which points can be inserted into the point set without leading to a change in the boundary of the point set. Now consider an interior point at a high scale, indicating that this point is not providing essential boundary information at that scale, then it is obvious that at a lower scale this point should also be interior. This implies that the interior at the lower scale should contain any interior at the higher scale, i.e., it should be increasing with decreasing scale.

One final demand follows from the fact that the point set is resulting from an irregular sampling of some *connected* object. Now, when sampling becomes infinitely dense, the boundary defined on S should be consistent with the boundary definition for connected point sets. Stated otherwise, it should approach the boundary of the connected object.

To summarize, a boundary graph to capture the shape of a point set should have the following properties:

1. *Scale*: A boundary graph should be part of a one-parameter family, with decreasing detail at decreasing scale, and the one-parameter family should allow for differential computation.
2. *Minimality*: At every scale the shape description should contain only the *essential* boundary information of S .
3. *Increasing interior*: The interior of the point

set should be increasing with decreasing scale.

4. *Consistency*: When the point set S is the result of sampling the connected set $X \subset \mathbb{R}^2$, the boundary as defined on S should approach the boundary of X when the sampling is infinitely dense on X .

As a potential boundary definition, we study in this paper the directed variant of the α -shape as proposed by Edelsbrunner et al. [4] and call it the α -graph. The α -graph is based on the α -hull, which is a generalization of the convex hull. An introduction to the work of Edelsbrunner et al. is given in section 2. Whereas the aforementioned reference [4] emphasizes the computational aspects of the α -shape, we study its use as a boundary definition, using the criteria as previously derived (section 4). As an example we show in section 5 how the α -graph could be used in the recognition of industrial objects. This work's relation to other work is presented in section 6. An important relation established in this paper pertains to the opening scale space from mathematical morphology [2].

2 The α -Hull and the α -Graph

This section gives an overview of the work of Edelsbrunner et al. on the α -hull and α -shape [3], [4].

We adopt the following notational conventions. Let X be a (closed or open) connected set in \mathbb{R}^2 . Then

$X^c \equiv$ the complement of X ,

$\bar{X} \equiv$ the set closure of X ,

$\overset{\circ}{X} \equiv$ the interior of X ,

$\partial X \equiv$ the boundary of X ,

$\mathcal{E}(\partial X) \equiv$ the ordered set of
(possibly curved) edges of ∂X ,

$\mathcal{V}(\partial X) \equiv$ the ordered set of vertices of ∂X .

Further, let G and G' be two graphs, and let x and y be two points in \mathbb{R}^2 , with $x \neq y$. Then

$\mathcal{E}(G) \equiv$ the edges of G ,

$\mathcal{V}(G) \equiv$ the vertices of G ,

$\{x, y\} \equiv$ the undirected edge connecting
 x and y ,

$(x, y) \equiv$ the edge directed from x to y .

The ordinary convex hull can be defined in several alternative ways. The α -hull is a generalization of the definition verbally given as "The convex hull of S is the intersection of all half-planes containing S ." In the definition of the α -hull the half-planes are replaced by generalized discs. As a consequence, the edges of the α -hull are circular arcs with curvature α , rather than straight line segments.

We first introduce the notion *generalized disc*. Let $B^*(c, r)$ denote the closed disc in \mathbb{R}^2 with center c and nonnegative radius r . Then the generalized disc $B(c, r)$ is defined as follows.

DEFINITION 1 (generalized disc).

$$B(c, r) \equiv \begin{cases} B^*(c, -r)^c & (r < 0), \\ B^*(c, r) & (r \geq 0). \end{cases}$$

Figure 1 shows examples of generalized discs of both positive and negative radii.

$C_\alpha(S)$, the set of α -centers, is the set of all centers of discs of radius $1/\alpha$ that have S as a subset.

DEFINITION 2 (α -centers).

$$C_\alpha(S) = \{x \in \mathbb{R}^2 \mid S \subseteq \overline{B(x, 1/\alpha)}\}.$$

Let X be an arbitrary (finite or connected) set in \mathbb{R}^2 , then the intersection of all closed generalized discs with varying centers $x \in X$ and fixed radius $1/\alpha$ is denoted by $M_\alpha(X)$, where we adopt the convention that the intersection over an empty set of generalized discs is equal to the entire plane.

DEFINITION 3 (intersection of discs).

$$M_\alpha(X) \equiv \begin{cases} \mathbb{R}^2 & (X = \emptyset), \\ \bigcap_{x \in X} \overline{B(x, 1/\alpha)} & (X \neq \emptyset). \end{cases}$$

The α -hull of S is the intersection of all closed generalized discs of radius $1/\alpha$ that contain all

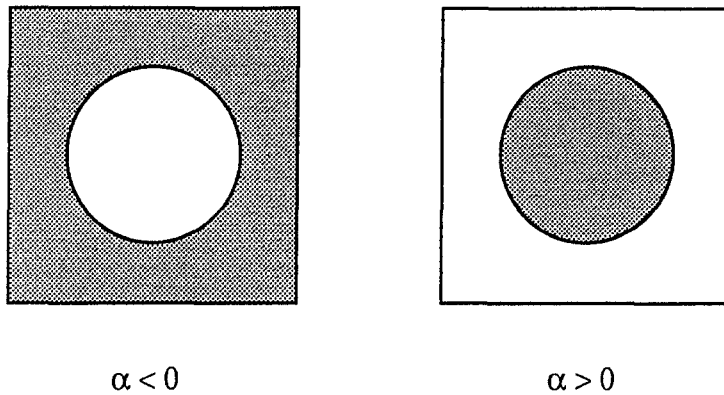


Fig. 1. Generalized discs for negative and positive α .

the points of S [4]. Thus the α -hull is given by the following definition.

DEFINITION 4 (α -hull).

$$H_\alpha(S) \equiv M_\alpha(C_\alpha(S)).$$

As in the limit case of infinite radius (i.e., α close to zero), the generalized disc becomes a half-plane; this definition also includes the common convex hull $H(S)$. In figure 2, $H_\alpha(S)$ is shown for an arbitrary negative and an arbitrary positive α ; the ordinary convex hull is also shown.

A point $s \in S$ is termed α -extreme in S if there exists a closed generalized disc of radius $1/\alpha$ such that s lies on its boundary and it contains all other points of S . The set of all α -extremes in S is denoted by $E_\alpha(S)$. It will be shown later that α -extremes are important for the boundary of the point set S .

DEFINITION 5 (α -extremes).

$$E_\alpha(S) \equiv \{s \in S \mid \exists c \in C_\alpha(S) : s \in \partial B(c, 1/\alpha)\}.$$

The set $M_\alpha(S)$ is nonempty if there exists at least one closed generalized disc containing the whole set S . This condition always holds for negative α , but for positive α it holds only if $1/\alpha$ is larger than the radius of the smallest enclosing circle. In the rest of the paper we always assume that this restriction on α holds, implying that $M_\alpha(S) \neq \emptyset$ for all α considered.

Now we define a relation between elements of $E_\alpha(S)$. Let $s, s' \in S$. Consider the (unique)

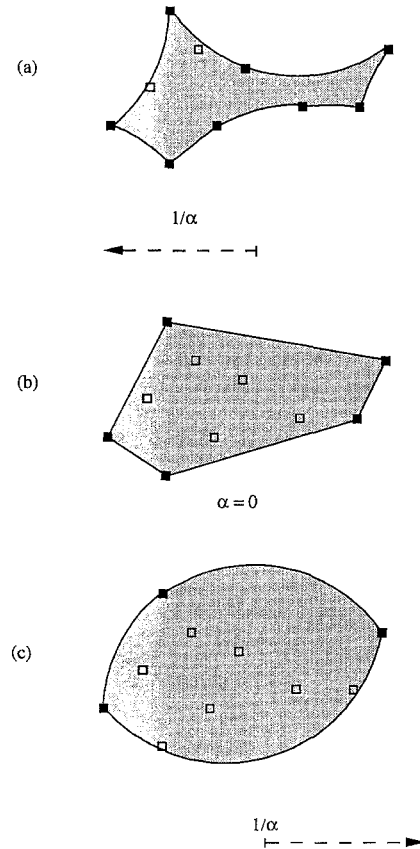


Fig. 2. α -Hull for an example set S : (a) negative α ; (c) positive α . Part (b) shows the ordinary convex hull. Note that the α -hull in (c) contains the α -hull in (b) and that, in turn, this α -hull contains the α -hull in (a). For ease of comparison, the positions of points falling in the interior of the α -hull are also shown.

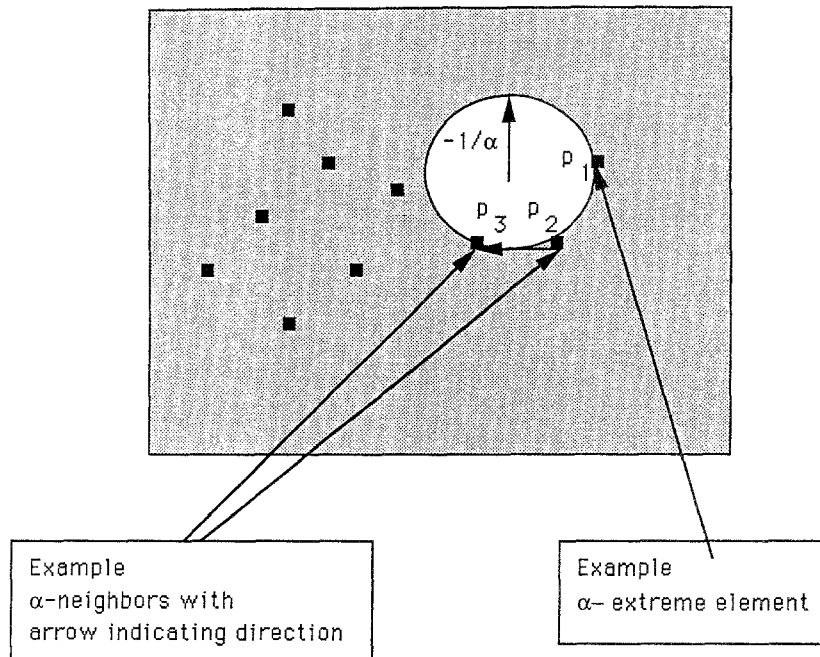


Fig. 3. The generalized disc of radius $1/\alpha$ with $\alpha < 0$ is one of the discs that make the points p_i elements of the α -extreme set of the point set. It also makes the pair (p_2, p_3) a pair of directed α -neighbors. Note that it is only an example of a pair, many of the other points are also α -neighbors or α -extremes. In fact, the elements of the directed pair (p_3, p_2) are also directed α -neighbors.

center c_α of the disc having radius $1/\alpha$, having s and s' on its boundary, and being positioned in the half-plane bounded by the directed edge (s, s') . Then the two elements of $E_\alpha(S)$ are identified as *directed neighbors* if the thus-defined disc contains all other points of the point set.

DEFINITION 6 (directed α -neighbors).

$$N_\alpha(S) \equiv \begin{cases} \{(s, s') \in S^2 \mid c_\alpha(s', s) \in C_\alpha(S)\} & (\alpha < 0), \\ \{(s, s') \in S^2 \mid c_\alpha(s, s') \in C_\alpha(S)\} & (\alpha \geq 0). \end{cases}$$

The two cases for $\alpha < 0$ and $\alpha \geq 0$ have been introduced to reach continuity in the limit case of $\alpha = 0$. The notions of α -extremes and α -neighbors are exemplified in figure 3.

The α -graph is the graph with vertices given by the α -extremes and directed edges given by the set of α -neighbors. Note that this is the directed variant of the α -shape as used in the reference. We introduced the term α -graph to make the distinction from the undirected α -shape explicit.

DEFINITION 7 (α -graph).

$$G_\alpha(S) \equiv \begin{cases} \mathcal{V}(G_\alpha(S)) = E_\alpha(S), \\ \mathcal{E}(G_\alpha(S)) = N_\alpha(S). \end{cases}$$

The α -graph is a directed graph with a special structure. Except for a finite set of values of α (the set $\text{TR}(S)$, which will be introduced in subsection 4.1), the faces of the graph are all cycles of the graph [4]. The faces bounded by a cycle with counterclockwise order are called *interior faces* [4].

Apart from interior faces, the α -graph contains isolated vertices, i.e., vertices $s \in E_\alpha(S)$ with $\text{deg}(s) = 0$ or bidirectional edges not lying in an interior face.

Now, consider the shape of the α -graph for increasing α . Let α_{\min} be half of the smallest distance between two elements in S . Then, for all $\alpha < -\alpha_{\min}$, all points are isolated. Increasing α will introduce bidirectional edges into the α -graph. Increasing α further will introduce, at some point, interior faces. These interior faces

merge with increasing α until the common convex hull is reached for $\alpha = 0$. For $\alpha = 0$ the region enclosed by the single interior face of the α -graph has maximum area. For larger positive α the area of this interior face is reduced. When $1/\alpha$ reaches the radius of the smallest disc enclosing the point set, the α -graph is reduced either to one triangular-shaped interior face or to one bidirectional edge. For higher values of α the α -graph is empty. The α -graph at different scales is illustrated in figure 4.

3 The Boundary and Interior of a Finite Point Set

The α -graph as defined in section 2 is a directed graph on a subset of the point set. As such, it might serve as a boundary definition for a finite point set. Thus we define the scale-dependent boundary of the finite point set $S \in \mathbb{R}^2$ as being the α -graph.

DEFINITION 8 (α -boundary).

$$\partial S_\alpha \equiv G_\alpha(S).$$

The scale-dependent interior of the finite point set is the region in \mathbb{R}^2 in which points can be inserted and not lead to a change in the boundary.

DEFINITION 9 (α -interior).

$$\overset{\circ}{S}_\alpha \equiv \{x \in \mathbb{R}^2 \mid G_\alpha(S \cup x) = G_\alpha(S)\}.$$

Note that the α -interior of S is not a finite point set. Thus the common definitions for connected point sets apply. However, the α -boundary of S does not equal the boundary of the α -interior, but it turns out to be closely related (see subsection 4.3).

4 Properties of the α -Graph

In section 1 we gave a set of criteria that a boundary definition should satisfy. In this section we systematically consider those criteria for the α -graph.

4.1 Minimality

A boundary of a finite point set should contain a minimal set of essential points describing the boundary; otherwise, information is lost or superfluous information is present. Because the boundary is defined as a function of scale, this necessary and sufficient set of points is also scale dependent. In this section we consider the minimality of the α -extremes of S as elements on the boundary of S .

Some basic properties of the operator M_α (Definition 3), needed in the rest of the paper, are given in the following proposition.

PROPOSITION 1.

$$M_\alpha(X \cup Y) = M_\alpha(X) \cap M_\alpha(Y), \quad (1)$$

$$\forall x \in X : M_\alpha(X) \subseteq \overline{B(x, 1/\alpha)}, \quad (2)$$

$$X \subseteq Y \Rightarrow M_\alpha(Y) \subseteq M_\alpha(X). \quad (3)$$

The operator M_α is directly related to the set C_α .

PROPOSITION 2.

$$C_\alpha(S) = M_\alpha(S).$$

Proof. Recall the definitions of the set of α -centers (Definition 2) and the definition of M_α (Definition 3). For $S = \emptyset$ the proposition clearly holds. For nonempty S we have

$$\begin{aligned} x \in C_\alpha(S) &\iff \forall s \in S : s \in \overline{B(x, 1/\alpha)} \\ &\iff \forall s \in S : x \in \overline{B(s, 1/\alpha)} \\ &\iff x \in \bigcap_{s \in S} \overline{B(s, 1/\alpha)} \\ &\iff x \in M_\alpha(S). \end{aligned}$$

This allows us to write the α -hull as a repeated application of the operator M_α .

PROPOSITION 3.

$$H_\alpha(S) = M_\alpha(M_\alpha(S)).$$

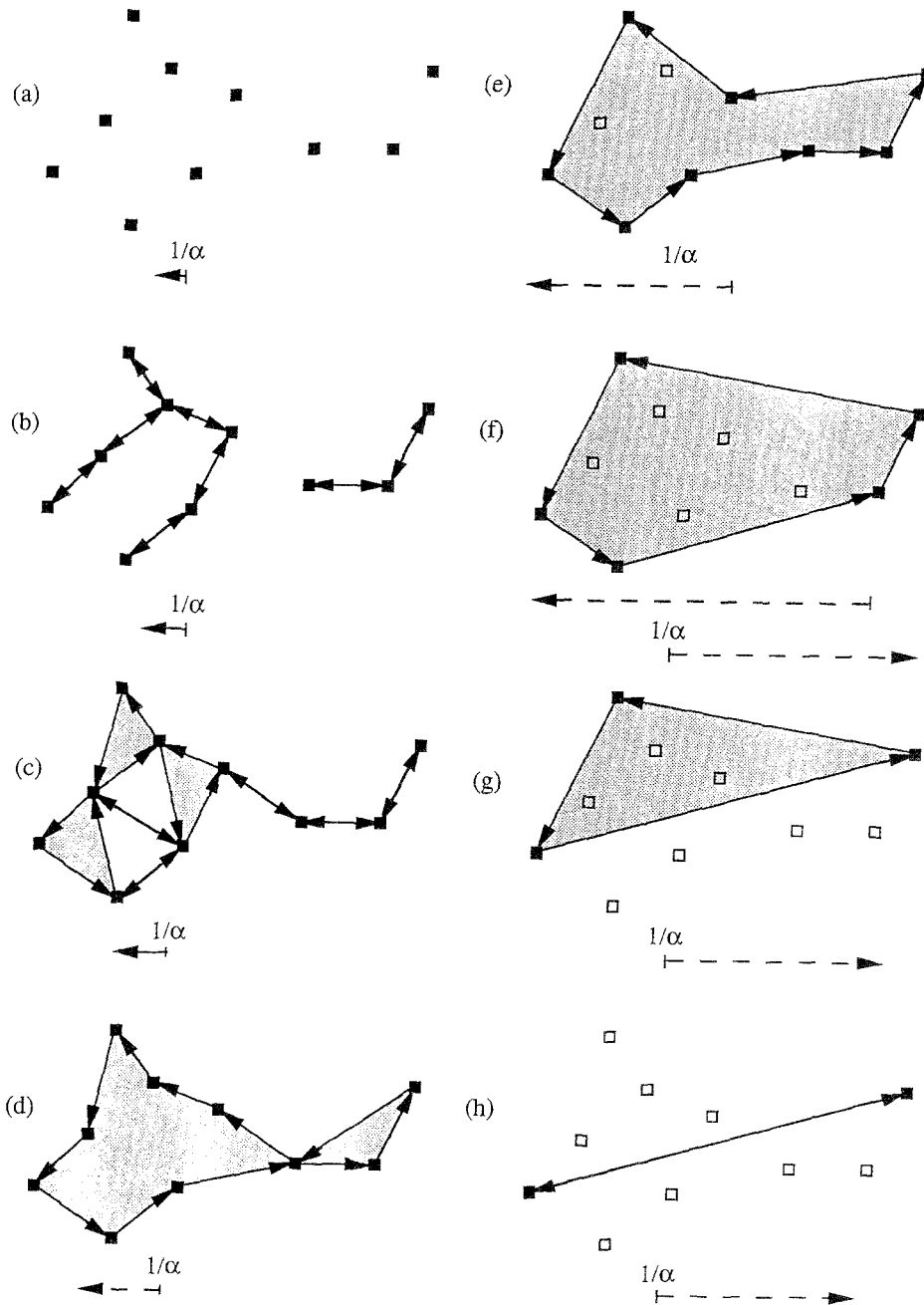


Fig. 4. Shape of the α -graph of point set S for increasing α . For α small enough, all points are isolated (a). Increasing α will introduce bidirectional edges into the α -graph (b). Increasing α further will introduce, at some point, interior faces (c). These interior faces merge with increasing α (d), (e) until the common convex hull is reached for $\alpha = 0$ (f). Note that the graph in (f) corresponds to both the negative and the positive values of α indicated, as well to $\alpha = 0$. Increasing α reduces the area of the interior face (g). At $\alpha_{\max} \approx 1/\rho(S)$, where $\rho(S)$ is the radius of the smallest disc containing S , the α -graph is reduced to one bidirectional edge (h). Note that (e), (f), and (g) correspond to the α -hulls in figure 2. The α -graph can be viewed as resulting from straightening the curved edges of the α -hull.

Proof. By definition, the α -hull is given by (Definition 4).

$$H_\alpha(S) = M_\alpha(C_\alpha(S)).$$

Hence from Proposition 2 we have

$$H_\alpha(S) = M_\alpha(M_\alpha(S)).$$

The set of α -extremes is important for the computation of $M_\alpha(S)$. In fact, the elements in S that are not in $E_\alpha(S)$ can be left out of the computation without altering the result.

PROPOSITION 4.

$$M_\alpha(S) = M_\alpha(E_\alpha(S)).$$

This proposition is proved by considering an arbitrary element of S not in $E_\alpha(S)$. This element does not contribute to the region $M_\alpha(S)$.

Proof. First, note that as $M_\alpha(S) \neq \emptyset$ we have that $E_\alpha(S) \neq \emptyset$. For the trivial case $E_\alpha(S) = S$, $M_\alpha(S) = M_\alpha(E_\alpha(S))$. Now let $s \in S \setminus E_\alpha(S)$. According to the definition of $E_\alpha(S)$ we have

$$\begin{aligned} s \in S \setminus E_\alpha(S) &\Rightarrow s \notin E_\alpha(S) \\ &\Rightarrow \forall c \in C_\alpha(S) : s \notin \partial B(c, 1/\alpha) \\ &\Rightarrow \forall c \in C_\alpha(S) : c \notin \partial B(s, 1/\alpha) \\ &\Rightarrow \forall c \in M_\alpha(S) : c \notin \partial B(s, 1/\alpha). \end{aligned}$$

This leads to three different cases for the relation of $M_\alpha(S)$ and $\partial B(s, 1/\alpha)$:

$$M_\alpha(S) \cap \overline{B(s, 1/\alpha)} = \emptyset, \quad (4)$$

$$M_\alpha(S) \subset \overline{B(s, 1/\alpha)}, \quad (5)$$

$$M_\alpha(S) \supset \overline{B(s, 1/\alpha)}. \quad (6)$$

Relation (2) of Proposition 1 and the fact that $M_\alpha(S) \neq \emptyset$ yield that (5) is the only possible case. But then we have that

$$M_\alpha(S \setminus \{s\}) = M_\alpha(S),$$

which implies that s is unnecessary for the computation of $M_\alpha(S)$. Because s is an arbitrary element of S , we conclude

$$M_\alpha(S) = M_\alpha(E_\alpha(S)).$$

It follows that the elements of $E_\alpha(S)$ are a sufficient point set for the computation of $M_\alpha(S)$. To prove that $E_\alpha(S)$ is also the necessary point set, we have to introduce the set $\text{TR}(S)$, which is based on concepts from computational geometry [11]. The definitions are given in the appendix. The set $\text{TR}(S)$ is the set of all reciprocals of the radii of the circles circumscribing the triangles in the closest- and furthest-point Delaunay triangulation of S . Figure 5 gives examples of two triangle radii. These finite sets of values in the continuous range of α will later turn out to be important in the description of the shape of S for varying scale.

Now we prove that the elements of $E_\alpha(S)$ in general form a necessary point set for the computation of M_α . In fact, when a proper subset of E_α is used in the computation, the result properly encloses the true result obtained when the whole set S is taken into consideration.

PROPOSITION 5.

$$\alpha \notin \text{TR}(S) \Rightarrow (\forall S' \subset E_\alpha(S) : M_\alpha(S) \subset M_\alpha(S')).$$

The proof considers all incomplete α -extreme sets S' . Then, by considering the contribution of the elements that have been left out to the edge of $M_\alpha(S)$ it is concluded that every point of the complete α -extreme set is necessary whenever α is not in $\text{TR}(S)$.

Proof. If $E_\alpha(S) = \emptyset$ or the set of α -extremes contains only one element, the proposition holds trivially. Therefore we consider the case for which $E_\alpha(S) \setminus S' \neq \emptyset$. Thus we have to establish only that elements of $E_\alpha(S)$ are necessary for the computation of $M_\alpha(S)$. Let $s \in E_\alpha(S) \setminus S'$, then from relation 3 of proposition 1 and from Proposition 4 we have

$$M_\alpha(S) \subseteq M_\alpha(E_\alpha(S) \setminus \{s\}).$$

Assume that $M_\alpha(S) = M_\alpha(S \setminus \{s\})$. From the fact that $s \in E_\alpha(S)$ and the definition of $E_\alpha(S)$ we have

$$\exists c \in C_\alpha(S) : s \in \partial B(c, 1/\alpha).$$

The equivalence of $C_\alpha(S)$ and $M_\alpha(S)$ (Proposition 2) leads to

$$\exists c \in m_\alpha(S) : c \in \partial B(s, 1/\alpha).$$

But this can be true only if

$$\exists c \in \partial M_\alpha(S) : c \in \partial B(s, 1/\alpha). \quad (7)$$

Every edge in $\partial M_\alpha(S)$ is centered at a unique element in S . Now, if part of the disc centered at s were an edge in $\partial M_\alpha(S)$, the boundary could not be the same as $\partial M_\alpha(S \setminus \{s\})$. Therefore the disc centered at s can intersect ∂M_α only at a vertex, i.e.,

$$\overline{B(s, 1/\alpha)} \cap \partial M_\alpha(S) \subseteq \mathcal{V}(\partial M_\alpha(S)).$$

Now, from (7) we have that this intersection cannot be empty. Let c be an element of this intersection. From the fact that c is not part of an edge, apart from the vertices of $\partial M_\alpha(S)$, we have

$$\exists s', s'' : s \neq s' \neq s'' \wedge s, s', s'' \in \partial B(c, 1/\alpha).$$

But, recalling the definition of N_α (Definition 6), we get

$$\{s, s'\}, \{s, s''\}, \{s', s''\} \in N_\alpha(S).$$

Because s, s' , and s'' are α -neighbors of one another, they form, depending on the sign of α , a triangle in the closest- or furthest-point Delaunay triangulation, denoted by DT_c and DT_f respectively. Using the fact that $s, s', s'' \in \partial B(c, 1/\alpha)$, we have

$$\begin{aligned} (\{s, s', s''\} \in DT_c(S) \vee \{s, s', s''\} \in DT_f(S)) \\ \wedge \rho(\{s, s', s''\}) = 1/|\alpha|, \end{aligned}$$

where $\rho(\{s, s', s''\})$ is the radius of the circle circumscribing the triangle with vertices s, s' and s'' . But then

$$\alpha \in \text{TR}(S).$$

Because this is exactly the restriction given on α , we conclude that

$$\alpha \notin \text{TR}(S) \Rightarrow (\forall S' \subset E_\alpha(S) : M_\alpha(S) \subset M_\alpha(S')).$$

The propositions immediately lead to the following theorem.

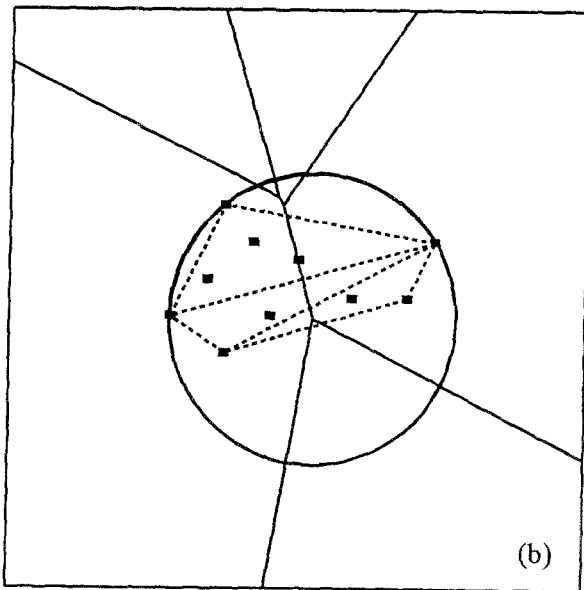
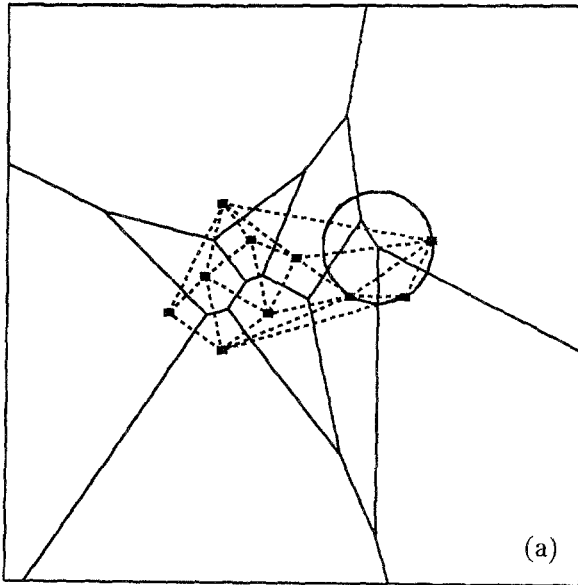


Fig. 5. (a) Closest-point Voronoi diagram of S (solid lines), and closest-point Delaunay triangulation (dashed lines). (b) For the same point set the furthest-point Voronoi diagram (solid lines) and furthest-point Delaunay triangulation (dashed lines). Examples of circles circumscribing the triangles in the triangulation are shown.

THEOREM 1. Let $S' \subseteq S$, Then

$$\begin{aligned} \alpha \notin \text{TR}(S) &\Rightarrow \\ (M_\alpha(S') = M_\alpha(S)) &\iff \\ E_\alpha(S) &\subseteq S'. \end{aligned}$$

Theorem 1 states that whenever α is not an element of $\text{TR}(S)$, the elements of $E_\alpha(S)$ are a sufficient and necessary point set for the computation of $M_\alpha(S)$ and therefore also for the computation of the α -graph. We conclude that the set $E_\alpha(S)$ contains all essential boundary information and further that it is the smallest set of points having this property. This also gives an immediate clue to the observation made in [4] that “different values of α give rise to hulls that have only in some sense *essential* points on their boundary.” Theorem 1 is illustrated in figure 6.

4.2 Scale

As stated in section 1, the boundary of a point set must be a function of a scale parameter whenever *a priori* information about the amount of detail present is absent. We demand that the boundary definition is a one-parameter family in which members show a decreasing amount of detail. Further, we demand that the family member at specific scale can be computed from any member at a higher scale.

The parameter α in the definition of the α -graph serves as a parameter of scale. The amount of detail can be expressed as the number of elements in the set of α -extremes. To identify α as a scale parameter, we prove the following theorem.

THEOREM 2.

$$\begin{aligned} \alpha \geq \alpha' &\Rightarrow (E_\alpha(S) \subseteq E_{\alpha'}(S)) \\ &\wedge E_\alpha(S) = E_\alpha(E_{\alpha'}(S)). \end{aligned}$$

Proof. Let $s \in S$. We first prove that elements are only removed from $E_\alpha(S)$ and are never introduced. From [4, Lemma 2] we have

$$\exists \alpha_{\max} : (s \in E_\alpha(S) \iff \alpha \leq \alpha_{\max}).$$

Thus

$$(\alpha \leq \alpha' \wedge s \in E_{\alpha'}(S)) \Rightarrow s \in E_\alpha(S).$$

Because s is an arbitrary element,

$$\alpha \geq \alpha' \Rightarrow E_\alpha(S) \subseteq E_{\alpha'}(S).$$

Combining this with Proposition 4 allows us to prove the differential computation part of the theorem:

$$\begin{aligned} \alpha \geq \alpha' &\Rightarrow E_\alpha(S) \subseteq E_{\alpha'}(S) \subseteq S \\ &\Rightarrow M_\alpha(E_{\alpha'}(S)) = M_\alpha(E_{\alpha'}(S)) \\ &\Rightarrow E_\alpha(E_{\alpha'}(S)) = E_\alpha(E_{\alpha'}(S)) \\ &\Rightarrow E_\alpha(S) = E_\alpha(E_{\alpha'}(S)). \end{aligned}$$

Theorem 2 identifies the parameter α as a genuine scale parameter on the set E_α . In this one-parameter family one can compute any member at a given scale from any member at a lower scale. An example of a point set at different scales is shown in figure 4.

4.3 Interior of a Finite Point Set

The α -interior $\overset{\circ}{S}$ of the finite point set S is defined as the region in \mathbb{R}^2 in which points can be inserted without leading to a change in the α -boundary of the point set (Definition 9). As a consequence of the fact that the α -boundary is a function of scale, the α -interior also depends on scale. In fact, the α -interior follows the rule “once not important at a high scale, never important at any lower scale.” In other words, the α -interior at a high scale is contained in any α -interior at a lower scale. It turns out that the α -interior of a finite point set is the interior of the α -hull. Note that the latter is the common definition of interior for connected sets.

PROPOSITION 6.

$$\overset{\circ}{S}_\alpha = \overset{\circ}{H}_\alpha(S).$$

This proposition is proved by first considering a point in the interior of the α -hull and proving that it does not lead to a change in the α -graph.

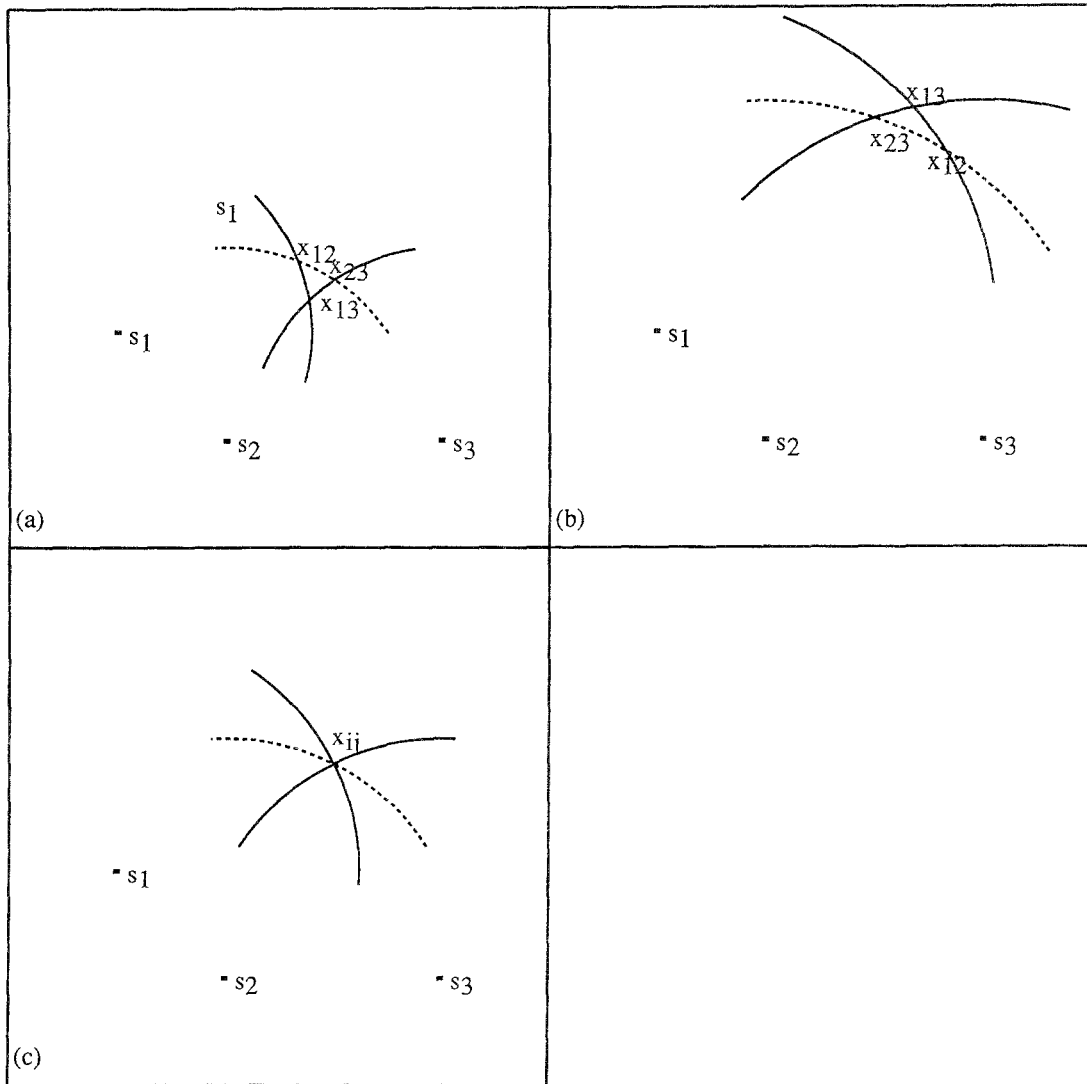


Fig. 6. To illustrate the α -extreme elements are sufficient and necessary for the computation of $M_\alpha(S)$, consider the three points s_1, s_2 , and s_3 . We concentrate on the intersections of the discs centered at the points s_i and to the left of edge (s_1, s_3) . The radii used in (a), (b), and (c) are $1/\alpha_a, 1/\alpha_b$, and $1/\alpha_c$, respectively. They are chosen such that $|1/\alpha_a| < |1/\alpha_b| = \rho(\{s_1, s_2, s_3\}) < |1/\alpha_c|$, where $\rho(\{s_1, s_2, s_3\})$ denotes the radius of the triangle with vertices s_1, s_2 , and s_3 . For positive α , s_2 is necessary in the computation of M_α for $\alpha = \alpha_b$ because then the edge from x_{12} to x_{23} , centered at s_2 (dashed curve), is part of ∂M_α . For $\alpha = \alpha_a$ the edge from x_{23} to x_{12} , centered at s_2 , is not part of ∂M_α and therefore s_1 and s_2 are sufficient for the computation of ∂M_α for $\alpha = \alpha_a$. The situation for which $\alpha = \alpha_c$ is precisely the transition between the situation for which α_a is an example and the situation for which α_b is an example. In this case all intersections x_{ij} coincide. For negative α the same results hold but with the roles of α_a and α_b reversed.

Then a point not in the interior is considered, and it is proved that it does lead to a change of the α -graph.

Proof. First, consider an element x in the in-

terior of the α -hull. Proposition 3 and the definition of the α -extremes (Definition 5) yield

$$\begin{aligned}
 x \in \overset{\circ}{H}_\alpha(S) &\Rightarrow \forall m \in M_\alpha(S) : x \in \overset{\circ}{B}(m, 1/\alpha) \\
 &\Rightarrow \forall m \in M_\alpha(S) : m \in \overset{\circ}{B}(x, 1/\alpha)
 \end{aligned}$$

$$\begin{aligned}
 &\Rightarrow M_\alpha(S) \subset \overset{\circ}{B}(x, 1/\alpha) \\
 &\Rightarrow x \notin E_\alpha(S \cup x) \\
 &\Rightarrow G_\alpha(S \cup x) = G_\alpha(S) \\
 &\Rightarrow x \in \overset{\circ}{S}_\alpha. \tag{8}
 \end{aligned}$$

Thus points in the interior of the α -hull do not lead to changes to the α -boundary of S . To prove that those are the only points with this property, let $x \in (H^\circ_\alpha(S))^c$ and $x \notin S$. Then, again using Proposition 3, we get

$$\begin{aligned}
 x \in (H^\circ_\alpha(S))^c &\Rightarrow \exists m \in M_\alpha(S) : x \notin \overset{\circ}{B}(m, 1/\alpha) \\
 &\Rightarrow \exists m \in M_\alpha(S) : m \notin \overset{\circ}{B}(x, 1/\alpha) \\
 &\Rightarrow M_\alpha(S) \not\subseteq \overline{B(x, 1/\alpha)}.
 \end{aligned}$$

It follows that either $\overline{B(x, 1/\alpha)} \subset M_\alpha(S)$ or $\overline{B(x, 1/\alpha)} \cap M_\alpha(S) \neq \emptyset$. Assume $\overline{B(x, 1/\alpha)} \subset M_\alpha(S)$, and let $s \in S$. Using relation (2) of Proposition 1, we get

$$\overline{B(x, 1/\alpha)} \subset M_\alpha(S) \subseteq \overline{B(x, 1/\alpha)}.$$

Because a closed disc can never be properly contained in a disc of the same radius, this cannot be correct and thus the assumption is not valid. As a consequence,

$$\begin{aligned}
 M_\alpha(S) \cap \overline{B(x, 1/\alpha)} &\neq \emptyset \\
 &\Rightarrow \exists m \in \partial B(m, 1/\alpha) : m \in M_\alpha(S) \\
 &\Rightarrow \exists m : S \subseteq \overline{B(x, 1/\alpha)} \\
 &\quad \wedge x \in \partial B(m, 1/\alpha) \\
 &\Rightarrow \exists m : (S \cup x) \subseteq \overline{B(m, 1/\alpha)} \\
 &\quad \wedge x \in \partial B(m, 1/\alpha) \\
 &\Rightarrow \exists m \in C_\alpha(S) : x \in \partial B(m, 1/\alpha) \\
 &\Rightarrow x \in E_\alpha(S \cup x).
 \end{aligned}$$

Because $x \notin S$, it follows that

$$G_\alpha(S \cup x) \neq G_\alpha(S)$$

and hence that

$$x \notin \overset{\circ}{S}_\alpha.$$

Combined with (8), this leads to the conclusion

$$\overset{\circ}{S}_\alpha = \overset{\circ}{H}_\alpha(S).$$

Note that because the edges of the α -hull have curvature α , in general the boundary of the α -interior is not the same as the α -boundary of the point set that has straight edges. Equivalence is reached only when either all elements of S are isolated in the α -graph or the edges of the α -hull are straight, i.e., the α -interior of S is the common convex hull ($\alpha = 0$).

To prove that the α -interior of the point set is increasing with increasing α , we use some concepts from mathematical morphology. From that theory (for an introduction see [5], [8], [14]) we will use the closing operation. In fact, the following property states that the α -hull can be written as the closing of S with a common non-generalized disc for $\alpha < 0$ or with the complement of a common disc for $\alpha > 0$. To abbreviate notation we use B_α to denote the generalized disc of radius $1/\alpha$ centered at the origin.

PROPOSITION 7.

$$H_\alpha(X) = X \bullet \overset{\circ}{B}_{-\alpha}.$$

Proof. First, we rewrite the operator M_α in terms of a morphological dilation:

$$\begin{aligned}
 M_\alpha(X) &= \cap_{x \in X} \overline{B(x, 1/\alpha)} \\
 &= (\cup_{x \in X} \overset{\circ}{B}(x, -1/\alpha))^c \\
 &= (X \oplus \overset{\circ}{B}_{-\alpha})^c.
 \end{aligned}$$

Now, using Proposition 3, we get

$$\begin{aligned}
 H_\alpha(X) &= ((X \oplus \overset{\circ}{B}_{-\alpha})^c \oplus \overset{\circ}{B}_{-\alpha})^c \\
 &= (X \oplus \overset{\circ}{B}_{-\alpha}) \ominus \overset{\circ}{B}_{-\alpha} \\
 &= X \bullet \overset{\circ}{B}_{-\alpha}.
 \end{aligned}$$

We use this relation to prove that the α -hull is increasing with increasing α (as was also observed in [4] but, was left unproved). The further consequences of this relation will be discussed in section 6.

THEOREM 3.

$$\alpha \leq \alpha' \Rightarrow \overset{\circ}{S}_\alpha \subseteq \overset{\circ}{S}_{\alpha'}.$$

This theorem is proved by considering two generalized discs $B_{\alpha'}$ and B_{α} , with $\alpha' \leq \alpha$. It will turn out that applying the closing operation to $B_{\alpha'}$ and using structuring element B_{α} will yield $B_{\alpha'}$. Combining this fact with a standard result from mathematical morphology relating closings in this situation proves the theorem.

Proof. A well-known property from mathematical morphology is

$$\begin{aligned} Y' \circ Y = Y' &\Rightarrow X \circ Y' \subseteq X \circ Y \subseteq X \\ &\subseteq X \bullet Y \subseteq X \bullet Y'. \end{aligned} \quad (9)$$

To apply this property in the context of generalized discs, we first have to prove

$$\alpha' \leq \alpha \Rightarrow B_{\alpha'} \circ B_{\alpha} = B_{\alpha'}.$$

We have three cases to consider, namely, $\alpha' \leq \alpha \leq 0$, $\alpha' \leq 0 \leq \alpha$, and $0 \leq \alpha' \leq \alpha$. In all cases it is easy to verify the following proposition:

$$B_{\alpha'} \ominus B_{\alpha} = B_{(1/\alpha' - 1/\alpha)}.$$

Note that for $\alpha' \leq \alpha \leq 0$ the generalized discs $B_{\alpha'}$ and B_{α} have negative radius but $B_{\alpha'} \ominus B_{\alpha}$ has positive radius. Using this way of writing the erosion, we have

$$\begin{aligned} B_{\alpha'} \circ B_{\alpha} &= (B_{\alpha'} \ominus B_{\alpha}) \oplus B_{\alpha} \\ &= B_{(1/\alpha' - 1/\alpha)} \oplus B_{\alpha} \\ &= B_{\alpha'}. \end{aligned}$$

Now, using Propositions 6 and 7 and applying (9) using generalized discs, we get

$$\begin{aligned} \alpha \leq \alpha' &\Rightarrow \overset{\circ}{B}_{-\alpha'} \circ \overset{\circ}{B}_{-\alpha} = \overset{\circ}{B}_{-\alpha'} \\ &\Rightarrow S \bullet \overset{\circ}{B}_{-\alpha} \subseteq S \bullet \overset{\circ}{B}_{-\alpha'} \\ &\Rightarrow H_{\alpha}(S) \subseteq H_{\alpha'}(S) \\ &\Rightarrow \overset{\circ}{H}_{\alpha}(S) \subseteq \overset{\circ}{H}_{\alpha'}(S) \\ &\Rightarrow \overset{\circ}{S}_{\alpha} \subseteq \overset{\circ}{S}_{\alpha'}. \end{aligned}$$

In this section it was proved that the interior of a point set as associated with the α -graph is increasing with respect to α in the sense that for $\alpha \leq \alpha'$ the interior at scale α is always contained in the interior at scale α' . An illustration of the theorem is found in figure 2.

4.4 Consistency

We assumed that the finite point set S was the result of an irregular sampling of some connected object X . Now, when sampling is infinitely dense on X , the α -boundary of S should approach the boundary of X , where the latter is the classical definition of boundary. The notion of α -extreme set can also be considered for the connected point set X . This α -extreme set can be found by intersecting the boundary of X with its α -hull.

PROPOSITION 8.

$$E_{\alpha}(X) = \partial X \cap \partial H_{\alpha}(X).$$

In this proof we first exclude elements in the interior of the α -hull, because it turns out that they can never be elements of the α -extreme set. Then we consider an element at the boundary of the α -hull, proving that it is an element of the α -extreme set.

Proof. First, consider an element in the interior of the α -hull. Using the equivalence of M_{α} and C_{α} (Proposition 2) and Proposition 3, we have

$$\begin{aligned} x \in \overset{\circ}{H}_{\alpha}(X) &\Rightarrow \forall c \in M_{\alpha}(X) : x \in \overset{\circ}{B}(c, 1/\alpha) \\ &\Rightarrow \forall c \in C_{\alpha}(X) : x \notin \partial B(c, 1/\alpha) \\ &\Rightarrow x \notin E_{\alpha}(X). \end{aligned}$$

So points in the interior are never α -extreme in S . Second, consider an element in the boundary of the α -hull

$$\begin{aligned} x \in \partial H_{\alpha}(X) &\Rightarrow \exists c \in M_{\alpha}(X) : x \in \partial B(c, 1/\alpha) \\ &\Rightarrow \exists c \in C_{\alpha}(X) : x \in \partial B(c, 1/\alpha) \\ &\Rightarrow x \in E_{\alpha}(X). \end{aligned}$$

Combining this with the fact that $X \subseteq H_{\alpha}(X)$ and the fact that $E_{\alpha}(X) \subseteq X$ (Definition 5), we conclude

$$\begin{aligned} x \in E_{\alpha}(X) &\iff x \in X \\ &\quad \wedge x \in \partial H_{\alpha}(X) \\ &\iff x \in \partial X \\ &\quad \wedge x \in \partial H_{\alpha}(X) \\ &\iff x \in \partial X \cap \partial H_{\alpha}(X). \end{aligned}$$

Now we prove the compatibility of the definition of the α -boundary of a finite point set, with the classical boundary definition of a connected point set. That is, we prove that for an increasingly dense sampling of a connected set, the α -graph approaches the boundary of the connected set. It turns out that to get consistency α should be infinitely small.

THEOREM 4.

$$\lim_{\alpha \downarrow -\infty} G_\alpha(X) = \partial X.$$

Proof. Taking the α -hull in the limit case of α going to minus infinity and using Proposition 8, we get

$$\lim_{\alpha \downarrow -\infty} H_\alpha(X) = \partial X \Rightarrow E_\alpha(X) = \partial X.$$

But then

$$\lim_{\alpha \downarrow -\infty} G_\alpha(X) = \partial X.$$

5 Application

As an example application, we show the use of the α -graph in the recognition of industrial objects. Such recognition is multiscale in the following sense. At low resolution the global size and orientation of the object are important. Those two aspects are given in a natural way by the α -graph with positive α . At a higher resolution (with negative α) the holes and finer shape details, such as concavities, are revealed. As α gets smaller, the α -graph breaks up into the principal components of the object. At the extreme end of the scale these components are the individual points. For all scales the set of α -extremes is minimal for that specific scale, resulting in the highest possible data reduction.

To demonstrate the performance, we selected an image of one of the objects used in a testbed for automated assembly (see figure 7).

The image was thresholded by using an automatic thresholding algorithm, resulting in a binary image. The binary image was sampled at reduced resolution (see figure 8(a)). In this

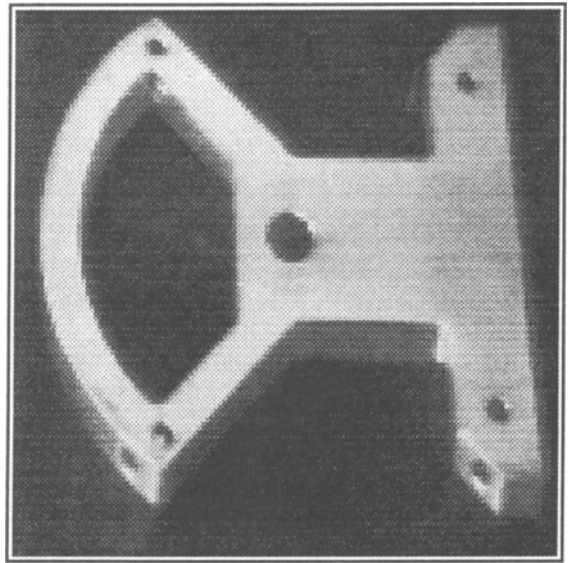


Fig. 7. Original image of an object used in a testbed for automated assembly.

case we used a regular grid, but this is not essential for the method. From there the α -graph was computed for different values of α . The ones showing details of interest were selected for display in this paper (see figure 8).

Note that in figure 8(a) the image is available in full detail as a collection of individual points. With increasing α the smallest holes disappear first; disappearance is related to the size of the hole with respect to the radius of the generalized disc. The larger holes also disappear, and for $\alpha = 0$ the convex hull is reached (figure 8(e)). Finally, for positive α (figure 8(f)) a small number of boundary points are identified, still yielding enough information to estimate the global position and orientation of the object.

6 Relation to Other Work

The work presented in this paper extends the work of [4]. In [4] the α -hull and α -shape (or α -graph) are defined, and from there the main results are efficient algorithms for computing them, including proof of their computational optimality. In our paper we emphasize the shape-descriptive power of the α -hull and α -

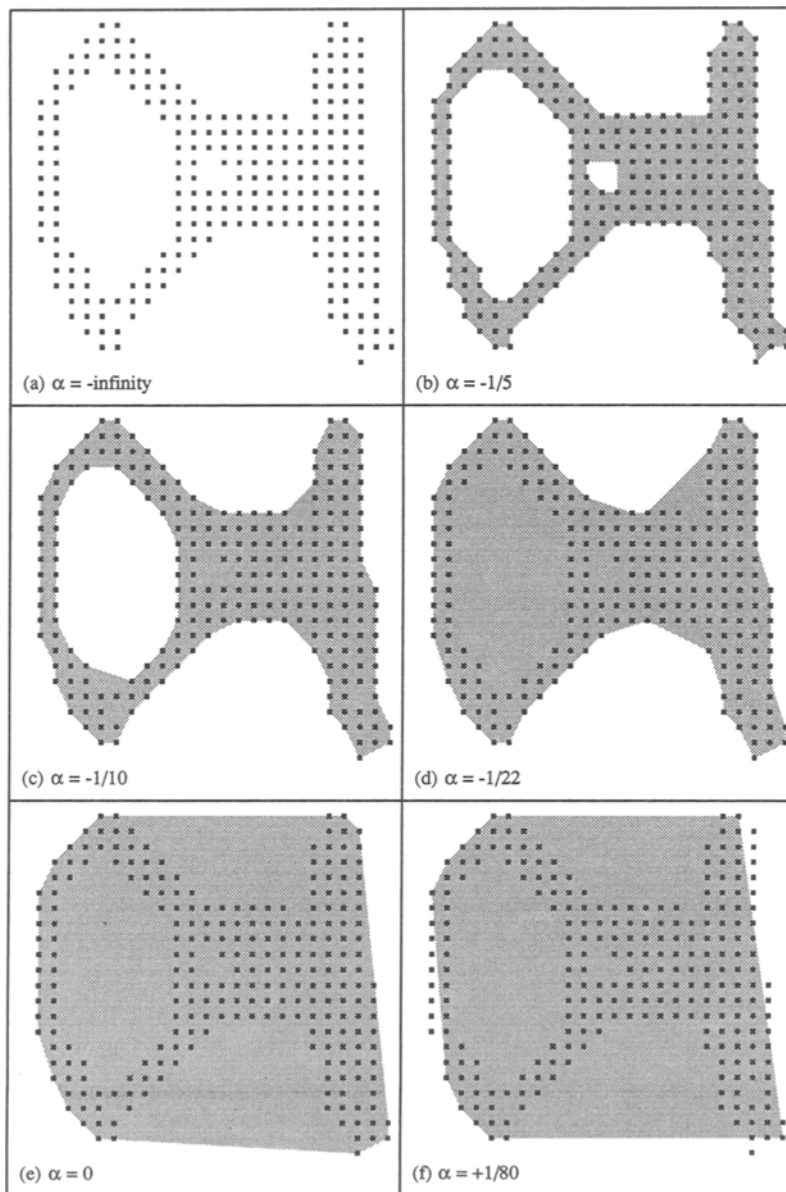


Fig. 8. α -graph of point set S resulting from thresholding and sampling the image of figure 7, for increasing α . The direction of the edges is used to define an enclosed region. Note that this region is different from the interior of the point set. Further, note that the extreme elements are the elements on the boundary of this enclosed region. For α sufficiently small the α -graph is equal to the point set S (a). In (b) the two largest holes in the objects are visible. Only one of these holes remains at a larger scale (c). A region containing no holes remains in (d). For $\alpha = 0$ the enclosed region is equal to the convex hull of S , and at this specific scale the region is equal to the interior of the point set. For the positive value of α shown in (f) a small set of extreme elements remains. However, these elements yield enough information to permit the global orientation and position of the object to be found.

graph. Several new properties of the α -hull and α -graph are derived. We further prove some properties that were observed in the reference

but were left unproved. The results in this paper identify the α -graph as a genuine boundary definition for finite point sets.

In [9] an alternative definition of the boundary of a finite point set is given. This definition is based on a measure of the density of point sets. The density measure is used to decompose the point set S into bounded components. These components are not necessarily convex. The main drawback of this definition is that the boundaries of the components may change if points are added to their interior. In our view, this contradicts the proper notion of interior (stated previously). An additional limitation of the method is the lack of a scale parameter (which cannot be missed in the shape description of finite point sets).

In [6] and [12] a distinction is made between the “internal” and “external” shapes of a finite point set. It is stated that “the external shape of a point set is exhibited by identifying the *essential* extreme points of the set and, among these, joining essential neighbors” and that “the internal shape of a point set is exhibited by identifying *essential* points of the set and, among these, joining *essential* neighbors.” In [6] and [12] the external shape is described by using the α -shape, where the internal shape is described by using a so-called β -shape. This β -shape is based on neighborhoods, with parameter β regulating the size of the neighborhood. The points x and y are called β -neighbors if their parameterized neighborhood is empty. The edges in the β -shape can be directed, much the same way as for the α -shape, by introducing the half-plane $W(x, y)$ into the definition of the β -neighborhood. In this way it is possible to make the α -graph a special case of the such-defined β -graph. However, because the internal shape depends on the definition of an empty neighborhood, it is not compatible with the shape of a connected object apart from points on the boundary of the point set, i.e., the extremes of the point set. Therefore only external shape is compatible with the shape of connected objects, and it should be referred to simply as *shape*. Internal shape in this context is better referred to as *spatial analysis* because the empty space between the points is a crucial part of the description.

It was proved in Proposition 7 that the α -hull is equivalent to the closing of X with a disc of radius $-1/\alpha$. Therefore from the duality of the

closing and the opening, the α -hull is the complement of the opening of X^c with the same disc as structuring element. This, in turn, is equivalent to the opening scale space as defined by Chen and Yan [2] for (a collection of) connected objects. For objects defined on a regular grid they use an approximation of a circular disc as structuring element.

The scale space in [2] is equivalent to the α -hull for negative values of α . The case of positive α is equivalent to the pseudoconvex hull as defined in [16]. So the α -hull can be viewed as a generalization of the opening scale space, combining it with the pseudoconvex hull into a one-parameter family.

In the identification of the α -hull as being the interior of the finite point set (Theorem 3) we used the following result from the theory of mathematical morphology [14]:

$$\begin{aligned} \alpha' \leq \alpha &\Rightarrow X \circ B_{\alpha'} \subseteq X \circ B_{\alpha} \\ &\subseteq X \subseteq X \bullet B_{\alpha} \subseteq X \bullet B_{\alpha'}. \end{aligned}$$

The scale-space aspects of the α -hull (and thus the opening scale space and the pseudoconvex hull) use only

$$X \subseteq X \bullet B_{\alpha} \subseteq X \bullet B_{\alpha'}.$$

This shows that the α -hull might, in fact, be part of some larger scale space. Note that for the finite point set S we have $S \circ B_{\alpha} = \emptyset$; thus this aspect of the α -hull is of interest only for (a collection of) connected sets. This aspect is the subject of further study.

For a *finite* point set S the α -hull is less suited for shape description than is the α -graph. The advantages of the α -graph over the α -hull stem from the following two aspects:

1. *Connectedness*: For a finite set a point in the α -hull is not necessarily connected to its α -neighbors. This is an undesired property of a shape descriptor. To appreciate this, consider two points s, s' and a generalized disc with negative radius just larger than half the distance between s and s' . All other points in S are assumed to be at such a distance from s and s' that they cannot be α -neighbors of s or s' . We can center a disc

at either side of the edge $\{s, s'\}$ in such a way that it has both points on its boundary and at the same time contains all other points. Thus the directed edges (s, s') and (s', s) will be in the α -graph. However, the edges can never be part of the intersection of the two generalized discs, and therefore they can never be part of the α -hull. In fact, in the α -hull only the endpoints s and s' of the edge remain. As a consequence, from the α -hull alone it can never be distinguished from the case in which there are two isolated points. (Calculating the α -hull for connected sets does not involve such problems).

2. *Discreteness*: The α -graph changes only at a finite collection of values of α . The set of all values is called the *shape spectrum* [3]. These values can be identified as being related to the distances between points in the nearest-neighbor graph as well as the elements of $\text{TR}(S)$, as introduced in subsection 4.1. Let α and α' be two different elements in the same interval between two successive elements of the shape spectrum. The only difference in the α -hulls corresponding to α and α' is the curvature of the edges between directed α -neighbors. For the finite set the shape at scale α is therefore not essentially different from the shape at scale α' . This again is an undesired property for a shape descriptor. (For connected sets the shape spectrum may consist of dense intervals in \mathbb{R} , and therefore the shape is continuously changing.)

These considerations give us reason to prefer the α -graph over the α -hull for determining the shape of a finite point set.

Note that the α -graph also provides a computationally efficient and consistent alternative to the use of an approximated disc when the opening scale space is defined for objects on a regular grid.

Another useful property from mathematical morphology is the following relation between dilation and closing [14]:

$$(X \bullet Y) \oplus Y = X \oplus Y.$$

Relating this to the operators M and H yields

$$M_\alpha(X) = M_\alpha(H_\alpha(X)).$$

Thus we have

$$H = MM, M = MH, H = HH, MH = HM.$$

It follows that the operators H and M form an Abelian group isomorphic to $\{1, -1\}_\times$, where H is the unit of the group. The consequences of this algebraic structure have not been studied yet.

7 Conclusion

In this section we discuss the suitability of the α -graph for shape description of a finite point set S on the basis of the demands raised in section 1.

1. *Scale*: From Theorem 2 it can be concluded that the amount of detail expressed as the number of vertices of the α -graph decreases monotonically with increasing α . The theorem further shows that this one-parameter family has the property that any member at scale $\alpha \geq \alpha'$ can be computed from the member at scale α' .
2. *Minimality*: The minimality of the description follows directly from Theorem 1, where the extreme elements of S are identified as being essential for the form of the α -graph. The minimality does not hold for the finite set of values $\alpha \in \text{TR}(S)$. This restriction could be removed easily by introducing a more elaborate definition of the α -graph (see [3]). We prefer the given definition because the values in $\text{TR}(S)$ indicate changes in the α -graph and are therefore important in the shape description for varying scales.
3. *Increasing interior*: The α -interior has been defined for point set S as the region in which additional elements can be positioned without altering the boundary. It is proved in proposition 6 that the interior of the α -hull defines this region. Further, it is proved in Theorem 3 that for $\alpha \leq \alpha'$ the interior at scale α is always contained in the interior at scale α' .
4. *Consistency*: When the finite point set S originates from a sampling of a connected object X in \mathbb{R}^2 , the α -boundary of S approaches

the boundary of X for α going to minus infinity and infinitely dense sampling (Theorem 4). From Proposition 8 the contour features length, orientation, and curvature can be defined in such a way that their values are consistent with features defined on the contour of X .

We conclude that the directed α -graph fulfills the four demands.

Apart from defining a boundary of a finite point set, the α -graph is important for shape description at multiple scales. Specifically, the identification of all essential α 's in the shape spectrum provides important shape information. To illustrate, we can compute a measure of local curvature as the change of orientation divided by the length of the edges. We can do so for every α in the shape spectrum and thus find the collection of critical curvature values associated with S . As an application we showed that the α -graph has potential in the multiscale analysis of industrial objects. At all scales the α -graph gives an optimal data reduction. A further application of α -graphs is found in the theory of curvature measurement from digitized contours [15], where it is used to compute the circular separability of two finite point sets.

Appendix: Notation from Computational Geometry

The α -graph is based on basic concepts from computational geometry (see, for example, [11]).

The generalized Voronoi polygon $V(P, Q)$ of arbitrary finite point sets P and Q in \mathbb{R}^2 is given by

$$V(P, Q) \equiv \{x \mid \forall p \in P, q \in Q \\ : d(x, p) \leq d(x, q)\},$$

where $d(x, y)$ denotes the distance from x to y .

Let S be a finite point set in \mathbb{R}^2 . Then the generalized Voronoi polygon is a generalization of the closest-point Voronoi polygon associated with $s \in S$ given by

$$V(s, S) = \{x \mid \forall s' \in S : d(x, s) \leq d(x, s')\}.$$

In an analogous way, the furthest-point Voronoi polygon associated with $s \in S$ is given by

$$V(S, s) = \{x \mid \forall s' \in S : d(x, s') \leq d(x, s)\}.$$

The union over the closest-point Voronoi polygons of all points in S provides a covering of the plane. This covering is called the *Closest-point Voronoi diagram*. The furthest-point Voronoi diagram is defined in the same way.

The region $V(p, q)$ is the half-plane containing p and bounded by the perpendicular bisector of (p, q) . It immediately follows that $V(P, Q)$ can be rewritten as

$$V(P, Q) = \bigcap_{p \in P, q \in Q} V(p, q).$$

From this formulation it can be concluded that $V(P, Q)$ is a filled (possibly unbounded) convex polygon.

Let $s, s' \in S$. Then the intersection of $V(s, S)$ and $V(s', S)$ is restricted to their (possibly empty) common boundary, with a similar result for the furthest-point Voronoi polygons. The points s and s' are commonly called *closest-point Voronoi neighbors* if the common boundary of their Voronoi polygons is nonempty.

The set of all closest-point Voronoi neighbor pairs in S is denoted by $N_c(S)$, where

$$N_c(S) \\ \equiv \{\{s, s'\} \in S^2 \mid \partial V(s, S) \cap \partial V(s', S) \neq \emptyset\}.$$

Similarly, s and s' are called *furthest-point Voronoi neighbors* if their Voronoi polygons have a nonempty common boundary. The set of all furthest-point Voronoi neighbor pairs in S is denoted by $N_f(S)$, where

$$N_f(S) \\ \equiv \{\{s, s'\} \in S^2 \mid \partial V(S, s) \cap \partial V(S, s') \neq \emptyset\}.$$

The graph with the elements of S as vertices and connecting each two closest-point Voronoi neighbors provides a triangulation of $H_0(S)$, called the *closest-point Delaunay triangulation* of S , here denoted by $\text{DT}_c(S)$. If the same notational convention as in section 2 are used, it is

given by

$$\text{DT}_c(S) \equiv \begin{cases} \mathcal{V}(\text{DT}_c(S)) = S, \\ \mathcal{E}(\text{DT}_c(S)) = N_c(S). \end{cases}$$

The furthest-point Delaunay triangulation is defined in a similar way, but because only vertices of $H_0(S)$ can be Voronoi neighbors, this triangulation of $H_0(S)$, denoted by $\text{DT}_f(S)$, is given by

$$\text{DT}_f(S) \equiv \begin{cases} \mathcal{V}(\text{DT}_f(S)) = \mathcal{V}(\partial H_0(S)), \\ \mathcal{E}(\text{DT}_f(S)) = N_f(S). \end{cases}$$

Examples of closest- and furthest-point Voronoi diagrams, together with the corresponding Delaunay triangulations, are shown in figure 5.

Acknowledgments

We greatly acknowledge the help of Rein van den Boomgaard with the results concerning mathematical morphology. We further thank one of the anonymous reviewers for detailed comments and for bringing the Abelian group structure of the operators to our attention.

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