University of Amsterdam
Theory of Computer Science

Division by Zero in Common Meadows

J.A. Bergstra
A. Ponse

Report TCS1410 June 2014

J.A. Bergstra

section Theory of Computer Science
Faculty of Science
University of Amsterdam

Science Park 904
1098 XH Amsterdam
the Netherlands

tel. +31 20 525.7591
e-mail: J.A.Bergstra@uva.nl

A. Ponse

section Theory of Computer Science
Faculty of Science
University of Amsterdam

Science Park 904
1098 XH Amsterdam
the Netherlands

tel. +31 20 525.7592
e-mail: A.Ponse@uva.nl

Theory of Computer Science Electronic Report Series

Division by Zero in Common Meadows

Jan A. Bergstra & Alban Ponse

Informatics Institute, University of Amsterdam
j.a.bergstra@uva.nl,   a.ponse@uva.nl

June 26, 2014

Abstract

Common meadows are fields expanded with a total inverse function. Division by zero produces an additional value denoted with **a** that propagates through all operations of the meadow signature (this additional value can be interpreted as an error element). We provide a basis theorem for so-called common cancellation meadows of characteristic zero, that is, common meadows of characteristic zero that admit a certain cancellation law.

Keywords and phrases: Meadow, common meadow, division by zero, additional value, abstract datatype.

Contents

1 Introduction

Elementary mathematics is uniformly taught around the world with a focus on natural numbers, integers, fractions, and fraction calculation. The mathematical basis of that part of mathematics seems to reside in the field of rational numbers. In elementary teaching material the incorporation of rational numbers in a field is usually not made explicit. This leaves open the possibility that some other abstract datatype or some alternative abstract datatype specification improves upon fields in providing a setting in which such parts of elementary mathematics can be formalized.

In this paper we will propose the signature for — and model class of — *common meadows* and we will provide a *loose* algebraic specification of common meadows by way of a set of equations. In the terminology of Broy and Wirsing [8, 12], the semantics of a loose algebraic specification S is given by the class of all models of S, that is, the semantic approach is not restricted to the isomorphism class of initial algebras.

A common meadow (using inversive notation) is an extension of a field equipped with a multiplicative inverse function $(...)^{-1}$ and an additional element $\mathbf{a}$ that serves as the inverse of zero and propagates through all operations. It should be noticed that the use of the constant $\mathbf{a}$ is a matter of convenience only because it merely constitutes a derived constant with defining equation $\mathbf{a} = 0^{-1}$. This implies that all uses of $\mathbf{a}$ can be removed from the story of common meadows (further comments can be found in Section 4: Concluding remarks).

The inverse function of a common meadow is not an involution because $(0^{-1})^{-1} = \mathbf{a}$. We will refer to meadows with zero-totalized inverse, that is, $0^{-1} = 0$, as *involutive meadows* because inverse becomes an involution. By default a "meadow" is assumed to be an involutive meadow.

The key distinction between meadows and fields, which we consider to be so important that it justifies a different name, is the presence of an operator symbol for inverse in the signature (inversive notation, see [3]) or for division (divisive notation, see [3]), where divisive notation x/y is defined as $x \cdot y^{-1}$. A major consequence is that fractions can be viewed as terms over the signature of (common) meadows. Another distinction between meadows and fields is that we do not require a meadow to satisfy the separation axiom $0 \neq 1$.

1.1 Common meadows versus involutive meadows

Involutive meadows, where instead of choosing $1/0 = \mathbf{a}$, one calculates with $1/0 = 0$, constitute a different solution to the question how to deal with the value of $1/0$ once the design decision has been made to work with the signature of meadows, that is to include a function name for inverse or for division (or both) in an extension of the syntax of fields. Involutive meadows feature a definite advantage over common meadows in that, by avoiding an extension of the domain with an additional value, theoretical work is very close to classical algebra of fields. This conservation property, conserving the domain, of involutive meadows has proven helpful for the development of theory about involutive meadows in [2, 1, 5, 3, 7, 6]; earlier and comparable work on the equational theory of fields was done by Komori [9] and Ono [11]. An equational axiomatization Md of involutive meadows is given in Table 1, where $^{-1}$ binds stronger than $\cdot$, which in turn binds stronger than $+$. From the axioms in Md the following identities are derivable:

$$
\begin{aligned}
0 \cdot x &= 0, & 0^{-1} &= 0,\\
x \cdot (-y) &= -(x \cdot y), & (-x)^{-1} &= -(x^{-1}),\\
-(-x) &= x, & (x \cdot y)^{-1} &= x^{-1} \cdot y^{-1}.
\end{aligned}
$$

$(x+y)+z = x+(y+z)$	$x \cdot y = y \cdot x$
$x+y = y+x$	$1 \cdot x = x$
$x+0 = x$	$x \cdot (y+z) = x \cdot y + x \cdot z$
$x+(-x) = 0$	$(x^{-1})^{-1} = x$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	$x \cdot (x \cdot x^{-1}) = x$

Table 1: The set Md of axioms for (involutive) meadows

Involutive *cancellation meadows* are involutive meadows in which the following cancellation law holds:

$$(x \neq 0 \wedge x \cdot y = x \cdot z) \rightarrow y = z. \tag{CL}$$

Involutive cancellation meadows form an important subclass of involutive meadows: in [1, Thm 3.1] it is shown that the axioms in Table 1 constitute a complete axiomatization of the equational theory of involutive cancellation meadows. We will use a consequence of this result in Section 3.

A definite disadvantage of involutive meadows against common meadows is that $1/0 = 0$ is quite remote from common intuitions regarding the partiality of division.

1.2 Motivating a preference for common meadows

Whether common meadows are to be preferred over involutive meadows depends on the applications one may have in mind. We envisage as an application area the development of revised foundations of elementary mathematics from a perspective of abstract datatypes, term rewriting, and mathematical logic. For that objective we consider common meadows to be the preferred option over involutive meadows. At the same time it can be acknowledged that a systematic investigation of involutive meadows constitutes a necessary stage in the development of a theory of common meadows by facilitating in a simplified setting the determination of results which might be obtained about common meadows. Indeed each result about involutive meadows seems to suggest a (properly adapted) counterpart in the setting of common meadows, while proving or disproving such counterparts is not an obvious matter.

2 Common Meadows

In this section we formally define "common meadows" by fixing their signature and providing an equational axiomatization. Then, we consider some conditional equations that follow from this axiomatization. Finally, we discuss some conditional laws that can be used to define an important subclass of common meadows.

2.1 Meadow signatures

The signature Σ_f^S of fields (and rings) contains a sort (domain) S, two constants 0, and 1, two two-place functions $+$ (addition) and $\cdot$ (multiplication) and the one-place function $-$ (minus) for the inverse of addition.

We write Σ^S_{md} for the signature of meadows in inversive notation:

$$\Sigma^S_{md} = \Sigma^S_f \cup \{\_^{-1} : S \to D\},$$

and we write $\Sigma^S_{md,\mathbf{a}}$ for the signature of meadows in inversive notation with a **a**-totalized inverse operator:

$$\Sigma^S_{md,\mathbf{a}} = \Sigma^S_{md} \cup \{\mathbf{a} \colon S\}.$$

The interpretation of **a** is called the additional value, and is often denoted $\hat{\mathbf{a}}$. Application of any function to the additional value returns that same value.

When the name of the carrier is fixed it need not be mentioned explicitly in a signature. Thus, with this convention in mind, Σ_{md} represents Σ^S_{md} and so on. If we want to make explicit that we consider terms over some signature Σ with variables in set X, we write $\Sigma(X)$.

Given a field several meadow signatures and meadows meadows can be connected with it. This will now be exemplified with the field $\mathbb{Q}$ of rational numbers. The following meadows are distinguished in this case:

$\mathbb{Q}_0$, the meadow of rational numbers with zero-totalized inverse: $\Sigma(\mathbb{Q}_0) = \Sigma^{\mathbb{Q}}_{md}$.

$\mathbb{Q}_{\mathbf{a}}$, the meadow of rational numbers with **a**-totalized inverse: $\Sigma(\mathbb{Q}_{\mathbf{a}}) = \Sigma^{\mathbb{Q}_a}_{md,\mathbf{a}}$. The additional value $\hat{\mathbf{a}}$ interpreting **a** has been taken outside $|\mathbb{Q}|$ so that $|\mathbb{Q}_{\hat{\mathbf{a}}}| = |\mathbb{Q}| \cup \{\hat{\mathbf{a}}\}$.

2.2 Axioms for common meadows

The axioms in Table 2 define the class (variety) of *common meadows*, where we adopt the convention that $\_^{-1}$ binds stronger than $\cdot$, which in turn binds stronger than $+$. Some comments: Axioms (15)−(17) take care of **a**'s propagation through all operations, and for the same reason, axioms (11) and (12) have their particular form. Axiom (4) is a variant of the common axiom on additional inverse, which also serves **a**'s propagation. Axioms (13) and (14) are further identities needed for manipulation of $(...)^{-1}$-expressions. Finally, axiom (10) is needed to reason with expressions of the form $0 \cdot t$.

The following propositions provide some more typical identities.

Proposition 2.2.1. *Equations that follow from* $\mathsf{Md}_{\mathbf{a}}$ *(see Table 2):*

$$0 \cdot 0 = 0, \tag{e1}$$
$$-0 = 0, \tag{e2}$$
$$0 \cdot x = 0 \cdot (-x), \tag{e3}$$
$$0 \cdot x \cdot x = 0 \cdot x, \tag{e4}$$
$$-(x \cdot y) = x \cdot (-y), \tag{e5}$$
$$(-1) \cdot x = -x, \tag{e6}$$
$$(-x)^{-1} = -(x^{-1}), \tag{e7}$$
$$x \cdot x^{-1} \cdot x^{-1} = x^{-1} \tag{e8}$$
$$-\mathbf{a} = \mathbf{a}, \tag{e9}$$
$$\mathbf{a}^{-1} = \mathbf{a}. \tag{e10}$$

$$(x + y) + z = x + (y + z) \tag{1}$$
$$x + y = y + x \tag{2}$$
$$x + 0 = x \tag{3}$$
$$x + (-x) = 0 \cdot x \tag{4}$$
$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \tag{5}$$
$$x \cdot y = y \cdot x \tag{6}$$
$$1 \cdot x = x \tag{7}$$
$$x \cdot (y + z) = x \cdot y + x \cdot z \tag{8}$$
$$-(-x) = x \tag{9}$$
$$0 \cdot (x \cdot y) = 0 \cdot (x + y) \tag{10}$$
$$(x^{-1})^{-1} = x + 0 \cdot x^{-1} \tag{11}$$
$$x \cdot x^{-1} = 1 + 0 \cdot x^{-1} \tag{12}$$
$$(x \cdot y)^{-1} = x^{-1} \cdot y^{-1} \tag{13}$$
$$1^{-1} = 1 \tag{14}$$
$$0^{-1} = \mathbf{a} \tag{15}$$
$$x + \mathbf{a} = \mathbf{a} \tag{16}$$
$$x \cdot \mathbf{a} = \mathbf{a} \tag{17}$$

Table 2: $\mathsf{Md}_{\mathbf{a}}$, a set of axioms for common meadows

Proof. Most derivations are trivial.

(e1). By axioms (3), (7), (8), and (2) we find $x = (1+0)\cdot x = x+0\cdot x = 0\cdot x+x$, hence $0\cdot 0 = 0\cdot 0+0 = 0$.

(e2). By axioms (3), (2), (4), and (e1) we find $-0 = -0 + 0 = 0 + -0 = 0 \cdot 0 = 0$.

(e3). By axioms (2), (4), and (9) we find $0 \cdot x = x + -x = -x + -(-x) = 0 \cdot (-x)$.

(e4). By axioms (10), (8), and (3), $0 \cdot x \cdot x = 0 \cdot x + 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x$.

(e5).

$$
\begin{aligned}
-(x \cdot y) &= -(x \cdot y) + 0 \cdot -(x \cdot y) && \text{by } x = x + 0 \cdot x \\
&= -(x \cdot y) + 0 \cdot (x \cdot y) && \text{by (e3)} \\
&= -(x \cdot y) + x \cdot (0 \cdot y) && \text{by axioms (5) and (6)} \\
&= -(x \cdot y) + x \cdot (y + (-y)) && \text{by axiom (4)} \\
&= -(x \cdot y) + (x \cdot y + x \cdot (-y)) && \text{by axiom (8)} \\
&= (-(x \cdot y) + x \cdot y) + x \cdot (-y) && \text{by axiom (1)} \\
&= 0 \cdot (x \cdot y) + x \cdot (-y) && \text{by axioms (2) and (4)} \\
&= 0 \cdot (x \cdot -y) + x \cdot (-y) && \text{by axioms (6) and (5), and (e3)} \\
&= x \cdot (-y). && \text{by } x = 0 \cdot x + x
\end{aligned}
$$

Thus, with axiom (9) it follows that $(-x) \cdot (-y) = x \cdot y$.

(e6). From (e5) with $y = 1$ we find $-x = -(x \cdot 1) = x \cdot (-1) = (-1) \cdot x$.

(e7). By axiom (12), $(-1) \cdot (-1)^{-1} = 1 + 0 \cdot (-1)^{-1}$, hence $(-1)^{-1} = (-1) + 0 \cdot (-1) \cdot (-1)^{-1} = (-1) + 0 \cdot (-1)^{-1}$. Now derive $1 = ((-1) \cdot (-1))^{-1} = (-1)^{-1} \cdot (-1)^{-1} = (-1)^{-1} \cdot ((-1) + 0 \cdot (-1)^{-1}) = (-1) \cdot (-1)^{-1} + 0 \cdot (-1)^{-1} \cdot (-1)^{-1} = (-1) \cdot (-1)^{-1} + 0 \cdot (1)^{-1} = (-1) \cdot (-1)^{-1}$, thus $(-1)^{-1} = -1$. Hence, $(-x)^{-1} = (-1 \cdot x)^{-1} = (-1)^{-1} \cdot x^{-1} = (-1) \cdot x^{-1} = -(x^{-1})$.

(e8). By axiom (12) and (e4), $x \cdot x^{-1} \cdot x^{-1} = (1 + 0 \cdot x^{-1}) \cdot x^{-1} = x^{-1} + 0 \cdot x^{-1} = x^{-1}$.

(e9). By (e5) and axioms (6) and (17), $-\mathbf{a} = -(\mathbf{a} \cdot 1) = \mathbf{a} \cdot (-1) = \mathbf{a}$.

(e10). By axioms (11) and (15) – (17), $\mathbf{a}^{-1} = (0^{-1})^{-1} = 0 + 0 \cdot \mathbf{a} = \mathbf{a}$. □

The next proposition establishes a generalization of a familiar identity concerning the addition of fractions.

Proposition 2.2.2. $\mathsf{Md}_\mathbf{a} \vdash x \cdot y^{-1} + u \cdot v^{-1} = (x \cdot v + u \cdot y) \cdot (y \cdot v)^{-1}$.

Proof. We first derive

$$\begin{aligned} x \cdot y \cdot y^{-1} &= x \cdot (1 + 0 \cdot y^{-1}) && \text{by axiom (12)} \\ &= x + 0 \cdot x \cdot y^{-1} \\ &= x + 0 \cdot x + 0 \cdot y^{-1} && \text{by axiom (10)} \\ &= x + 0 \cdot y^{-1}. \end{aligned} \tag{18}$$

Hence,

$$\begin{aligned} (x \cdot v + u \cdot y) \cdot (y \cdot v)^{-1} &= x \cdot y^{-1} \cdot v \cdot v^{-1} + u \cdot v^{-1} \cdot y \cdot y^{-1} \\ &= (x \cdot y^{-1} + 0 \cdot v^{-1}) + (u \cdot v^{-1} + 0 \cdot y^{-1}) && \text{by (18)} \\ &= (x \cdot y^{-1} + 0 \cdot y^{-1}) + (u \cdot v^{-1} + 0 \cdot v^{-1}) \\ &= x \cdot y^{-1} + u \cdot v^{-1}. \end{aligned}$$ □

Proposition 2.2.3. *For each term t over $\Sigma_{\mathsf{Md},\mathbf{a}}(X)$ with variables in X, either $\mathsf{Md}_\mathbf{a} \vdash t = \mathbf{a}$, or there exist terms r_1, r_2 over $\Sigma_f(X)$ such that $\mathsf{Md}_\mathbf{a} \vdash t = r_1 \cdot r_2^{-1}$ and $\mathsf{VAR}(t) = \mathsf{VAR}(r_1) \cup \mathsf{VAR}(r_2)$.*

Proof. By induction on the structure of t, where the $\mathsf{VAR}(t)$-property follows easily in each case.

If $t \in \{0, 1, x, \mathbf{a}\}$ this follows trivially (for the first three cases we need $1^{-1} = 1$).

Case $t \equiv t_1 + t_2$. If at least one of t_i is provably equal to $\mathbf{a}$, axiom (16) applies, and otherwise Proposition 2.2.2 applies.

Case $t \equiv t_1 \cdot t_2$. Trivial.

Case $t \equiv -t_1$. Proposition 2.2.1 applies (either (e9) or (e5)).

Case $t \equiv t_1^{-1}$. If t_1 is provably equal to $\mathbf{a}$, Proposition 2.2.1.e10 applies; otherwise there exist by induction $r_i \in \Sigma_f(X)$ such that $\mathsf{Md}_\mathbf{a} \vdash t_1 = r_1 \cdot r_2^{-1}$. Now derive $t_1^{-1} = r_1^{-1} \cdot (r_2^{-1})^{-1} = r_1^{-1} \cdot (r_2 + 0 \cdot r_2^{-1}) = r_2 \cdot r_1^{-1} + 0 \cdot r_1^{-1} + 0 \cdot r_2^{-1} = r_2 \cdot r_1^{-1} + 0 \cdot r_2^{-1}$ and apply Proposition 2.2.2. □

Proposition 2.2.4. *For each term t over $\Sigma_f(X)$, $\mathsf{Md}_\mathbf{a} \vdash 0 \cdot t = 0 \cdot \sum_{x \in \mathsf{VAR}(t)} x$, where $\sum_{x \in \emptyset} x = 0$.*

Proof. By induction on the structure of t, where axiom (10) covers the multiplicative case. □

$x \neq \mathbf{a} \;\rightarrow\; 0 \cdot x = 0$	Normal Value Law	(NVL)
$x^{-1} = \mathbf{a} \;\rightarrow\; 0 \cdot x = x$	Additional Value Law	(AVL)
$x \neq 0 \wedge x \neq \mathbf{a} \;\rightarrow\; 0 \cdot x^{-1} = 0$	Normal Inverse Value Law	(NIVL)
$x \neq 0 \wedge x \neq \mathbf{a} \;\rightarrow\; x \cdot x^{-1} = 1$	Common Inverse Law	(CIL)

Table 3: Some conditional laws for common meadows

2.3 Conditional equations

In this section we discuss a number of conditional equations that are useful for common meadows, and we start off with a few that follow directly from $\mathsf{Md}_{\mathbf{a}}$.

Proposition 2.3.1. *Conditional equations that follow from* $\mathsf{Md}_{\mathbf{a}}$ *(see Table 2):*

$$x \cdot y = 1 \rightarrow 0 \cdot y = 0, \tag{ce1}$$

$$x \cdot y = 1 \rightarrow x^{-1} = y, \tag{ce2}$$

$$0 \cdot x = 0 \cdot y \rightarrow 0 \cdot (x \cdot y) = 0 \cdot x, \tag{ce3}$$

$$0 \cdot x \cdot y = 0 \rightarrow 0 \cdot x = 0, \tag{ce4}$$

$$0 \cdot (x + y) = 0 \rightarrow 0 \cdot x = 0, \tag{ce5}$$

$$0 \cdot x^{-1} = 0 \rightarrow 0 \cdot x = 0, \tag{ce6}$$

$$0 \cdot x = \mathbf{a} \rightarrow x = \mathbf{a}. \tag{ce7}$$

Proof. Most derivations are trivial.

(ce1). By (e4), $0 \cdot x \cdot x = 0 \cdot x$, so $0 \cdot x \cdot y = 0 \cdot x \cdot y \cdot y = 0 \cdot x \cdot y + 0 \cdot y \cdot y = (0 \cdot x + 0 \cdot y) \cdot y$, and hence by assumption, $0 = 0 \cdot 1 = 0 \cdot x \cdot y = (0 \cdot x + 0 \cdot y) \cdot y = 0 \cdot x \cdot y + 0 \cdot y \cdot y = 0 + 0 \cdot y = 0 \cdot y$.

(ce2). By assumption and axioms (13) and (14), $x^{-1} \cdot y^{-1} = 1$, and thus by (ce1), $0 \cdot x^{-1} = 0$, so by axiom (12), $y = (1 + 0 \cdot x^{-1}) \cdot y = (x \cdot x^{-1}) \cdot y = (x \cdot y) \cdot x^{-1} = x^{-1}$.

(ce3). By assumption and axioms (10) and (8), $0 \cdot (x \cdot y) = 0 \cdot x + 0 \cdot y = 0 \cdot x + 0 \cdot x = 0 \cdot x$.

(ce4). By assumption, $0 \cdot x = 0 \cdot x + 0 \cdot x \cdot y = x \cdot (0 + 0 \cdot y) = 0 \cdot (x \cdot y) = 0$.

(ce5). Apply axiom (10) to (ce4).

(ce6). By axiom (12) and assumption, $x \cdot x^{-1} = 1 + 0 \cdot x^{-1} = 1$, so by (ce1), $0 \cdot x = 0$.

(ce7). By $x = x + 0 \cdot x$ and assumption, $x = x + \mathbf{a} = \mathbf{a}$. $\square$

Note that (ce1) and (ce2) immediately imply

$$x \cdot y = 1 \rightarrow 0 \cdot x^{-1} = 0.$$

In Table 3 we define various conditional laws that we will use later on to single out certain classes of common meadows: the Normal Value Law (NVL), the Additional Value Law (AVL), the Normal Inverse Value Law (NIVL), and the Common Inverse Law (CIL). Here we use the adjective "normal" to express that values different from **a** (more precisely, the interpretation of **a**) are at stake.

Proposition 2.3.2.

1. $\mathsf{Md}_\mathbf{a} + \mathsf{NVL} \vdash (x \cdot y = \mathbf{a} \wedge x \neq \mathbf{a}) \to y = \mathbf{a}$,
2. $\mathsf{Md}_\mathbf{a} + \mathsf{NVL} \vdash x^{-1} \neq \mathbf{a} \to 0 \cdot x = 0$,
3. $\mathsf{Md}_\mathbf{a} + \mathsf{NVL} + \mathsf{AVL} \vdash \mathsf{CIL}$,
4. $\mathsf{Md}_\mathbf{a} + \mathsf{CIL} \vdash \mathsf{NVL}$,
5. $\mathsf{Md}_\mathbf{a} + \mathsf{CIL} \vdash \mathsf{AVL}$,
6. $\mathsf{Md}_\mathbf{a} + \mathsf{NIVL} \vdash \mathsf{CIL}$,
7. $\mathsf{Md}_\mathbf{a} + \mathsf{CIL} \vdash \mathsf{NIVL}$,

Proof. 1. By NVL, $x \neq \mathbf{a} \to 0 \cdot x = 0$, so $0 \cdot y = (0 \cdot x) \cdot y = 0 \cdot (x \cdot y) = 0 \cdot \mathbf{a} = \mathbf{a}$ and hence $y = (1 + 0) \cdot y = y + 0 \cdot y = y + \mathbf{a} = \mathbf{a}$.

2. By NVL, $0 \cdot x^{-1} = 0$ and hence by axiom (12), $x \cdot x^{-1} = 1$ and by (ce1), $0 \cdot x = 0$.

3. From $x \neq \mathbf{a}$ we find $0 \cdot x = 0$. There are two cases: $x^{-1} = \mathbf{a}$ which implies by AVL that $x = 0$ contradicting the assumptions of CIL, and $x^{-1} \neq \mathbf{a}$ which implies by NVL that $0 \cdot x^{-1} = 0$, and this implies $x \cdot x^{-1} = 1$ by axiom (12).

4. Assume that $x \neq \mathbf{a}$. If $x = 0$ then also $0 \cdot x = 0$. If $x \neq 0$ then by CIL, $0 = 0 \cdot 1 = 0 \cdot x \cdot x^{-1}$, so $0 \cdot x = 0$ by (ce1).

5. We distinguish three cases: $x = 0$, $x = \mathbf{a}$, and $x \neq 0 \wedge x \neq \mathbf{a}$. In the first two cases it immediately follows that $0 \cdot x = x$. In the last case it follows by CIL that $x \cdot x \cdot x^{-1} = x$, so $x^{-1} = \mathbf{a}$ implies $x = \mathbf{a}$, and thus $x = 0 \cdot x$.

6. Apply axiom (12).

7. If $x \neq 0$ and $x \neq \mathbf{a}$, then by CIL, $x \cdot x^{-1} = 1$, hence $0 \cdot x \cdot x^{-1} = 0$, so by axiom (12), $0 \cdot (1 + 0 \cdot x^{-1}) = 0$, and thus $0 \cdot x^{-1} = 0$. □

3 Models and model classes

In this section we define "common cancellation meadows" as common meadows that satisfy the so-called "inverse cancellation law", a law that is equivalent with the Common Inverse Law CIL. Then, we provide a basis theorem for common cancellation meadows of characteristic zero.

3.1 Common cancellation meadows

In [1, Thm 3.1] we prove a generic basis theorem that implies that the axioms in Table 1 constitute a complete axiomatization of the equational theory of the involutive cancellation meadows (over signature Σ_{md}). The cancellation law used in that result (that is, CL in Section 1.1) has various equivalent versions, and a particular one is $x \neq 0 \to x \cdot x^{-1} = 1$, a version that is close to CIL.

Below we define common cancellation meadows, using a cancellation law that is equivalent with CIL, but first we establish a correspondence between models of $\mathsf{Md}_\mathbf{a}$ + NVL + AVL and involutive cancellation meadows.

Proposition 3.1.1.

1. *Every field can be extended with an additional value* $\hat{\mathbf{a}}$ *and subsequently it can be expanded with a constant* $\mathbf{a}$ *and an inverse function in such a way that the equations of common meadows as well as* NVL *and* AVL *are satisfied, where the interpretation of* $\mathbf{a}$ *is* $\hat{\mathbf{a}}$.

2. *A model of* $\mathsf{Md}_{\mathbf{a}} + \mathsf{NVL} + \mathsf{AVL}$ *extends a field with an additional value* $\hat{\mathbf{a}}$ *(the interpretation of* $\mathbf{a}$ *) and expands it with the* $\mathbf{a}$*-totalized inverse.*

Proof. Statement 1 follows immediately. To prove 2, consider the substructure of elements b of the domain that satisfy $0 \cdot b = 0$. Only $\hat{\mathbf{a}}$ is outside this subset. For b with $0 \cdot b = 0$ we must check that $0 \cdot b^{-1} = 0$ unless $b = 0$. To see this distinguish two cases: $b^{-1} = \mathbf{a}$ (which implies $b = 0$ with help of AVL), and $b^{-1} \neq \mathbf{a}$ which implies $0 \cdot b^{-1} = 0$ by NVL. □

As a consequence, we find the following result.

Theorem 3.1.2. *The models of* $\mathsf{Md}_{\mathbf{a}} + \mathsf{NVL} + \mathsf{AVL}$ *that satisfy* $0 \neq 1$ *are in one-to-one correspondence with the involutive cancellation meadows satisfying* Md *(see Table 1) and* $0 \neq 1$.

Proof. An involutive cancellation meadow can be expanded to a model of $\mathsf{Md}_{\mathbf{a}} + \mathsf{NVL} + \mathsf{AVL}$ by extending its domain with a constant $\hat{\mathbf{a}}$ and an inverse function in such a way that the equations of common meadows as well as NVL and AVL are satisfied, where the interpretation of $\mathbf{a}$ is $\hat{\mathbf{a}}$ (cf. Proposition 3.1.1.1).

Conversely, given a model $\mathbb{M}$ of $\mathsf{Md}_{\mathbf{a}} + \mathsf{NVL} + \mathsf{AVL}$, we construct a cancellation meadow $\mathbb{M}'$ as follows: $|\mathbb{M}'| = |\mathbb{M}| \setminus \{\hat{\mathbf{a}}\}$ with $\hat{\mathbf{a}}$ the interpretation of $\mathbf{a}$, and $0^{-1} = 0$. We find by NVL that $0 \cdot x = 0$ and by NIVL (thus by NVL + AVL, cf. Proposition 2.3.2.[3,7]) that $x \neq 0 \to x \cdot x^{-1} = 1$, which shows that $\mathbb{M}'$ is a cancellation meadow. □

We define a *common cancellation meadow* as a common meadow that satisfies the following *inverse cancellation law* (ICL):

$$(x \neq 0 \wedge x \neq \mathbf{a} \wedge x^{-1} \cdot y = x^{-1} \cdot z) \to y = z. \qquad \text{(ICL)}$$

The class CCM of common cancellation meadows is axiomatized by $\mathsf{Md}_{\mathbf{a}} + \mathsf{CIL}$ in Table 2 and Table 3, respectively. In combination with $\mathsf{Md}_{\mathbf{a}}$, the laws ICL and CIL are equivalent: first, $\mathsf{Md}_{\mathbf{a}} + \mathsf{ICL} \vdash \mathsf{CIL}$ because

$$(x \neq 0 \wedge x \neq \mathbf{a}) \stackrel{(e8)}{\to} (x \neq 0 \wedge x \neq \mathbf{a} \wedge x^{-1} \cdot x \cdot x^{-1} = x^{-1} \cdot 1) \stackrel{\mathsf{ICL}}{\to} x \cdot x^{-1} = 1.$$

Conversely, $\mathsf{Md}_{\mathbf{a}} + \mathsf{CIL} \vdash \mathsf{ICL}$:

$$(x \neq 0 \wedge x \neq \mathbf{a} \wedge x^{-1} \cdot y = x^{-1} \cdot z) \to x \cdot x^{-1} \cdot y = x \cdot x^{-1} \cdot z \stackrel{\mathsf{CIL}}{\to} y = z.$$

3.2 A basis theorem for common cancellation meadows of characteristic zero

As in our paper [2], we use numerals $\underline{n}$ and the axiom scheme C_0 defined in Table 4 to single out common cancellation meadows of characteristic zero. In this section we prove that $\mathsf{Md}_{\mathbf{a}} + \mathsf{C}_0$ constitutes an axiomatization for common cancellation meadows of characteristic zero. In [2, Cor. 2.7] we prove that $\mathsf{Md} + \mathsf{C}_0$ (for Md see Table 1) constitutes an axiomatization for involutive cancellation meadows of characteristic zero. We define CCM_0 as the class of common cancellation meadows of characteristic zero.

We further write $\dfrac{t}{r}$ (and sometimes t/r in plain text) for $t \cdot r^{-1}$.

$\underline{n+1} \cdot (\underline{n+1})^{-1} = 1$	$(n \in \mathbb{N})$	(C_0)
$\underline{0} = 0$	(axioms for	
$\underline{1} = 1$	numerals,	
$\underline{n+1} = \underline{n} + 1$	$n \in \mathbb{N}$ and $n \geq 1$)	

Table 4: C_0, the set of axioms for meadows of characteristic zero, and axioms for numerals

Theorem 3.2.1. $\mathsf{Md}_{\mathbf{a}} + \mathsf{C}_0$ *is a basis for the equational theory of* CCM_0.

Proof. Soundness holds by definition of CCM_0.

Assume $\mathrm{CCM}_0 \models t = r$ and $\mathrm{CCM}_0 \models t = \mathbf{a}$. Then, by axioms (15) – (17) and identities (e9) – (e10), t and r are provably equal to $\mathbf{a}$, that is, $\mathsf{Md}_{\mathbf{a}} \vdash t = r$.

Assume $\mathrm{CCM}_0 \models t = r$ and $\mathrm{CCM}_0 \not\models t = \mathbf{a}$. By Proposition 2.2.3 we can bring t in the form t_1/t_2 and r in the form r_1/r_2 with t_i, r_i terms over $\Sigma_f(X)$, thus

$$\mathrm{CCM}_0 \models \frac{t_1}{t_2} = \frac{r_1}{r_2}. \tag{19}$$

We will first argue that (19) implies that the following three equations are valid in CCM_0:

$$0 \cdot t_2^{-1} = 0 \cdot r_2^{-1}, \tag{20}$$

$$0 \cdot t_1 + 0 \cdot t_2 = 0 \cdot r_1 + 0 \cdot r_2, \tag{21}$$

$$t_2 \cdot r_2 \cdot (t_1 \cdot r_2 + -r_1 \cdot t_2) + 0 \cdot t_2^{-1} + 0 \cdot r_2^{-1} = 0 \cdot t_1 + 0 \cdot t_2^{-1} + 0 \cdot r_1 + 0 \cdot r_2^{-1}. \tag{22}$$

Ad (20). Assume this is not the case, then there exists a common cancellation meadow $\mathbb{M} \in \mathrm{CCM}_0$ and an interpretation of the variables in t_2 and r_2 such that one of t_2^{-1} and r_2^{-1} is interpreted as $\hat{\mathbf{a}}$ (the interpretation of $\mathbf{a}$), and the other is not. This contradicts (19).

Ad (21). This equation characterizes that t_1/t_2 and r_1/r_2 contain the same variables, and is related to Proposition 2.2.4. Assume this is not the case, say t_1 and/or t_2 contains a variable x that does not occur in r_1 and r_2. Since $\mathrm{CCM}_0 \not\models r_1/r_2 = \mathbf{a}$, there is an instance of r_i's variables, say $\overline{r_i}$ such that $\mathrm{CCM}_0 \models \overline{r_1}/\overline{r_2} \neq \mathbf{a}$. But then x can be instiantiated with $\mathbf{a}$, which contradicts (19).

Ad (22). It follows from (19) that in (22) both the lefthand-side and the righthand-side equal zero in all involutive cancellation meadows. By Theorem 3.1.2 we find $\mathrm{CCM} \models$ (22), and hence $\mathrm{CCM}_0 \models$ (22).

We now argue that (20) – (22) are derivable from $\mathsf{Md}_{\mathbf{a}} + \mathsf{C}_0$, and that from those (19) is derivable from $\mathsf{Md}_{\mathbf{a}} + \mathsf{C}_0$.

Ad (20). The statement $\mathrm{CCM}_0 \models 0 \cdot t_2^{-1} = 0 \cdot r_2^{-1}$ is equivalent with the statement that t_2 and r_2 have the same zeros in the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and in the algebraic closure F_p of each prime field $\overline{F_p}$. We may assume that the gcd of t_2's coefficients is 1, and similar for r_2: if not, then $t_2 = k \cdot t'$ with t' a polynomial with that property, and since k is a fixed numeral, we find $0 \cdot k = 0$ (also in fields with a characteristic that is a factor of k), and hence $0 \cdot t_2 = 0 \cdot t'$. We can apply [10, (Ch.IV, Corollary 2.4)]: because t_2 and r_2 are polynomials in $\Sigma_f(\mathsf{VAR}(t_2, r_2))$ with the property that they have the same zeros and that the gcd of their coefficients is 1, they have equal factorization in primitive polynomials. So, in common cancellation meadows of characteristic zero (thus, models in CCM_0), each such factor of t_2 is one of r_2, and vice versa. Application of (e4) (that is, $0 \cdot x \cdot x = 0 \cdot x$) then yields

$$\mathsf{Md}_{\mathbf{a}} + \mathsf{C}_0 \vdash 0 \cdot t_2^{-1} = 0 \cdot r_2^{-1}. \tag{23}$$

Ad (21). From Proposition 2.2.4 and validity of (21) it follows that

$$\mathsf{Md}_{\mathbf{a}} \vdash 0 \cdot t_1 + 0 \cdot t_2 = 0 \cdot \textstyle\sum_{x \in \mathsf{VAR}(t_1/t_2)} x = 0 \cdot \sum_{x \in \mathsf{VAR}(r_1/r_2)} x = 0 \cdot r_1 + 0 \cdot r_2. \tag{24}$$

Ad (22). We first derive

$$\begin{aligned} \mathsf{Md}_{\mathbf{a}} \vdash 0 \cdot t_1 + 0 \cdot t_2^{-1} &= 0 \cdot t_1 + 0 \cdot (1 + 0 \cdot t_2^{-1}) \\ &= 0 \cdot t_1 + 0 \cdot t_2 \cdot t_2^{-1} && \text{with axiom (12)} \\ &= 0 \cdot t_1 + 0 \cdot t_2 + 0 \cdot t_2^{-1}, \end{aligned}$$

and in a similar way one derives $\mathsf{Md}_{\mathbf{a}} \vdash 0 \cdot r_1 + 0 \cdot r_2^{-1} = 0 \cdot r_1 + 0 \cdot r_2 + 0 \cdot r_2^{-1}$. Hence, we find with (23) and (24) that

$$\mathsf{Md}_{\mathbf{a}} + \mathsf{C}_0 \vdash 0 \cdot t_1 + 0 \cdot t_2^{-1} = (0 \cdot t_1 + 0 \cdot t_2^{-1}) + (0 \cdot r_1 + 0 \cdot r_2^{-1}) \tag{25}$$

$$= 0 \cdot r_1 + 0 \cdot r_2^{-1}. \tag{26}$$

From $\mathrm{CCM}_0 \models (22)$ it follows from the completeness result on the class of involutive meadows of characteristic zero (see [2, Cor. 2.7]) that $\mathsf{Md} + \mathsf{C}_0 \vdash (22)$, and hence $\mathsf{Md}_{\mathbf{a}} + \mathsf{C}_0 \vdash (22)$.

We now show the derivability of $t_1/t_2 = r_1/r_2$. Multiplying both sides of (22) with $(t_2 \cdot r_2)^{-1} \cdot (t_2 \cdot r_2)^{-1}$ implies by (e8), $0 \cdot x + 0 \cdot x = 0 \cdot x$, and (e4) that

$$\mathsf{Md}_{\mathbf{a}} + \mathsf{C}_0 \vdash (t_2 \cdot r_2)^{-1} \cdot (t_1 \cdot r_2 + -r_1 \cdot t_2) + 0 \cdot t_2^{-1} + 0 \cdot r_2^{-1} = 0 \cdot t_1 + 0 \cdot t_2^{-1} + 0 \cdot r_1 + 0 \cdot r_2^{-1},$$

which implies by Proposition 2.2.2 that

$$\mathsf{Md}_{\mathbf{a}} + \mathsf{C}_0 \vdash \frac{t_1}{t_2} + \frac{-r_1}{r_2} + 0 \cdot t_2^{-1} + 0 \cdot r_2^{-1} = 0 \cdot t_1 + 0 \cdot t_2^{-1} + 0 \cdot r_1 + 0 \cdot r_2^{-1},$$

and thus

$$\mathsf{Md}_{\mathbf{a}} + \mathsf{C}_0 \vdash \frac{t_1}{t_2} + \frac{-r_1}{r_2} + 0 \cdot t_1 + 0 \cdot t_2^{-1} + 0 \cdot r_1 + 0 \cdot r_2^{-1} = 0 \cdot t_1 + 0 \cdot t_2^{-1} + 0 \cdot r_1 + 0 \cdot r_2^{-1}, \tag{27}$$

and hence

$$\begin{aligned} \mathsf{Md}_{\mathbf{a}} + \mathsf{C}_0 \vdash \frac{t_1}{t_2} &= \frac{t_1}{t_2} + 0 \cdot t_1 + 0 \cdot t_2^{-1} \\ &= \frac{t_1}{t_2} + 0 \cdot t_1 + 0 \cdot t_2^{-1} + 0 \cdot r_1 + 0 \cdot r_2^{-1} && \text{by (25)} \\ &= \frac{t_1}{t_2} + (\frac{r_1}{r_2} + \frac{-r_1}{r_2}) + 0 \cdot t_1 + 0 \cdot t_2^{-1} + 0 \cdot r_1 + 0 \cdot r_2^{-1} \\ &= (\frac{t_1}{t_2} + \frac{-r_1}{r_2}) + \frac{r_1}{r_2} + 0 \cdot t_1 + 0 \cdot t_2^{-1} + 0 \cdot r_1 + 0 \cdot r_2^{-1} \\ &= \frac{r_1}{r_2} + 0 \cdot t_1 + 0 \cdot t_2^{-1} + 0 \cdot r_1 + 0 \cdot r_2^{-1} && \text{by (27)} \\ &= \frac{r_1}{r_2} + 0 \cdot r_1 + 0 \cdot r_2^{-1} && \text{by (26)} \\ &= \frac{r_1}{r_2}. \end{aligned}$$

□

4 Concluding remarks

Open question. It is an open question is whether there exists a basis result for the equational theory of CCM. We notice that in [4] a basis result for one-totalized non-involutive cancellation meadows is provided, where the multiplicative inverse of 0 is 1 and cancellation is defined as usual (that is, by the cancellation law CL in Section 1.1).

Common intuitions. As a variant of the involutive and common meadows, *partial meadows* are defined in [3]. Common meadows are motivated as being the most intuitive modelling of a totalized inverse function to the best of our knowledge. As stated in Section 1 (Introduction), the use of the constant $\mathbf{a}$ is a matter of convenience only because it merely constitutes a derived constant with defining equation $\mathbf{a} = 0^{-1}$, which implies that all uses of $\mathbf{a}$ can be removed.[1] We notice that considering $\mathbf{a} = 0^{-1}$ as an error-value supports the intuition for the equations of $\mathsf{Md}_{\mathbf{a}}$.

Quasi-cancellation meadows of characteristic zero. Following Theorem 3.2.1, a common meadow of characteristic zero can alternatively be defined as a structure that satisfies all equations true of all common cancellation meadows of characteristic zero. We write CM for the class of all common meadows, and CM_0 for all common meadows of characteristic zero.

With this alternative definition in mind, we define a *common quasi-cancellation meadow of characteristic zero* as a common meadow that satisfies all conditional equations which are true of all common cancellation meadows of characteristic zero. We write CQCM_0 for the class of all common quasi-cancellation meadows of characteristic zero.

It is easy to show that CQCM_0 is strictly larger than CCM_0. To see this one extends the signature of common meadows with a new constant c. Let $L_{ccm,0}$ be the set of conditional equations true of all structures in CCM_0. We consider the initial algebra of $L_{ccm,0}$ in the signature extended with c. Now neither $L_{ccm,0} \vdash c = \mathbf{a}$ can hold (because c might be interpreted as say 1), nor $L_{ccm,0} \vdash 0 \cdot c = 0$ can hold (otherwise $L_{ccm,0} \vdash 0 = 0 \cdot \mathbf{a} = \mathbf{a}$ would hold). For that reason in the initial algebra of $L_{ccm,0}$ in the extended signature interprets c as an entity e in such a way that neither $c = \mathbf{a}$ nor $0 \cdot c = 0$ is satisfied. For that reason c will be interpreted by a new entity that refutes CIL.

CM_0 is strictly larger than CQCM_0. To see this let $E_{ccm,0}$ denote the set of equations valid in all common cancellation models of characteristic zero. Again we add an extra constant b to the signature of common meadows. Consider the initial algebra I of $E_{ccm,0} + b^{-1} = \mathbf{a}$ in the extended signature. In I the interpretation of b is a new object because it cannot be proven equal to 0 and not to $\mathbf{a}$ and not to any other closed term over the signature of common meadows. Now we transform $E_{ccm,0} + b^{-1} = \mathbf{a}$ into its set of closed consequences $E^{cl,b}_{ccm,0}$ over the extended signature. We claim that $b = 0 \cdot b$ cannot be proven from $E_{ccm,0} + b^{-1} = \mathbf{a}$. If that were the case at some stage in the derivation an $\mathbf{a}$ must appear from which it follows that $b = \mathbf{a}$ is provable as well, because $\mathbf{a}$ is propagated by all operations. But that cannot be the case as we have already concluded that b differs from $\mathbf{a}$ in the initial algebra of $E^{cl,b}_{ccm,0}$.

However, at this stage we do not know the answers to the following two questions:

- Is there a finite equational basis for the class CM_0 of common meadows of characteristic zero?
- Is there a finite conditional equational basis for the class CQCM_0 of common quasi-cancellation meadows of characteristic zero?

[1] We notice that $0 = 1 + (-1)$, from which it follows that 0 can also be considered a derived constant over a reduced signature. Nevertheless, the removal of 0 from the signature of fields is usually not considered helpful.

Acknowledgement

We thank Bas Edixhoven (Leiden University) for helpful comments concerning the proof of Theorem 3.2.1, including his suggestion to use [10] for this proof.

References

[1] Bergstra, J.A., Bethke, I., and Ponse, A. (2013). Cancellation meadows: a generic basis theorem and some applications. *The Computer Journal*, 56(1):3–14, doi:10.1093/comjnl/bsx147.

[2] Bergstra, J.A., Bethke, I., and Ponse, A. (2013). Equations for formally real meadows. arXiv:1310.5011v3 [math.RA, cs.LO], 18 Oct 2013, this version (v3): 11 Feb 2014.

[3] Bergstra, J.A. and Middelburg, C.A. (2011). Inversive meadows and divisive meadows. *Journal of Applied Logic*, 9(3):203–220.

[4] Bergstra, J.A. and Middelburg, C.A. (2014). Division by zero in non-involutive meadows. arXiv:1406.2092v1 [math.RA], 9 Jun 2014.

[5] Bergstra, J.A. and Ponse, A. (2013). Signed meadow valued probability mass functions. arXiv:1307.5173v1 [math.LO], 19 Jul 2013.

[6] Bergstra, J.A. and Tucker, J.V. (2007). The rational numbers as an abstract data type. *Journal of the ACM*, 54(2), Article 7 (25 pages).

[7] Bethke, I. and Rodenburg, P.H. (2010). The initial meadows. *Journal of Symbolic Logic*, 75(3):888–895.

[8] Broy, M. and Wirsing, M. (1981). On the algebraic specification of nondeterministic programming languages. In Astesiano, E. and Böhm, C. (eds), *Proc. CAAP81*, Genova, Italy. LNCS 112, pp. 162–179. Springer.

[9] Komori, Y. (1975). Free algebras over all fields and pseudo-fields. Report 10, pp. 9–15, Faculty of Science, Shizuoka University.

[10] Lang, S. (2002). *Algebra*. Graduate Texts in Mathematics, Vol. 211, third revised edition. Springer.

[11] Ono, H. (1983). Equational theories and universal theories of fields. *Journal of the Mathematical Society of Japan*, 35(2):289–306.

[12] Wirsing, M. (1990). Algebraic specifications. In J. van Leeuwen (ed.), *Handbook of Theoretical Computer Science. Volume B: Formal models and semantics*, North-Holland, 1990, pp 675–788.

Electronic Reports Series of section Theory of Computer Science

Within this series the following reports appeared.

[TCS1407v3] J.A. Bergstra, *Four Complete Datatype Defining Rewrite Systems for an Abstract Datatype of Natural Numbers (version 3),* section Theory of Computer Science - University of Amsterdam, 2014.

[TCS1409] J.A. Bergstra and A. Ponse, *Three Datatype Defining Rewrite Systems for Datatypes of Integers each extending a Datatype of Naturals,* section Theory of Computer Science - University of Amsterdam, 2014.

[TCS1406v3] J.A. Bergstra, *Bitcoin and Islamic Finance (version 3),* section Theory of Computer Science - University of Amsterdam, 2014.

[TCS1407v2] J.A. Bergstra, *Four Complete Datatype Defining Rewrite Systems for an Abstract Datatype of Natural Numbers (version 2),* section Theory of Computer Science - University of Amsterdam, 2014.

[TCS1408] J.A. Bergstra, *Bitcoin: Informational Money en het Einde van Gewoon Geld,* section Theory of Computer Science - University of Amsterdam, 2014.

[TCS1407] J.A. Bergstra, *Four Complete Datatype Defining Rewrite Systems for an Abstract Datatype of Natural Numbers,* section Theory of Computer Science - University of Amsterdam, 2014.

[TCS1406v2] J.A. Bergstra, *Bitcoin and Islamic Finance (version 2),* section Theory of Computer Science - University of Amsterdam, 2014.

[TCS1406] J.A. Bergstra, *Bitcoin and Islamic Finance,* section Theory of Computer Science - University of Amsterdam, 2014.

[TCS1405] J.A. Bergstra, *Rekenen in een Conservatieve Schrapwet Weide,* section Theory of Computer Science - University of Amsterdam, 2014.

[TCS1404] J.A. Bergstra, *Division by Zero and Abstract Data Types,* section Theory of Computer Science - University of Amsterdam, 2014.

[TCS1403] J.A. Bergstra, I. Bethke, and A. Ponse, *Equations for Formally Real Meadows,* section Theory of Computer Science - University of Amsterdam, 2014.

[TCS1402] J.A. Bergstra and W.P. Weijland, *Bitcoin, a Money-like Informational Commodity,* section Theory of Computer Science - University of Amsterdam, 2014.

[TCS1401] J.A. Bergstra, *Bitcoin, een "money-like informational commodity",* section Theory of Computer Science - University of Amsterdam, 2014.

[TCS1301] B. Diertens, *The Refined Function-Behaviour-Structure Framework,* section Theory of Computer Science - University of Amsterdam, 2013.

[TCS1202] B. Diertens, *From Functions to Object-Orientation by Abstraction,* section Theory of Computer Science - University of Amsterdam, 2012.

[TCS1201] B. Diertens, *Concurrent Models for Object Execution,* section Theory of Computer Science - University of Amsterdam, 2012.

[TCS1102] B. Diertens, *Communicating Concurrent Functions,* section Theory of Computer Science - University of Amsterdam, 2011.

[TCS1101] B. Diertens, *Concurrent Models for Function Execution,* section Theory of Computer Science - University of Amsterdam, 2011.

[TCS1001] B. Diertens, *On Object-Orientation,* section Theory of Computer Science - University of Amsterdam, 2010.

Within former series (PRG) the following reports appeared.

[PRG0914] J.A. Bergstra and C.A. Middelburg, *Autosolvability of Halting Problem Instances for Instruction Sequences,* Programming Research Group - University of Amsterdam, 2009.

[PRG0913] J.A. Bergstra and C.A. Middelburg, *Functional Units for Natural Numbers,* Programming Research Group - University of Amsterdam, 2009.

[PRG0912] J.A. Bergstra and C.A. Middelburg, *Instruction Sequence Processing Operators,* Programming Research Group - University of Amsterdam, 2009.

[PRG0911] J.A. Bergstra and C.A. Middelburg, *Partial Komori Fields and Imperative Komori Fields,* Programming Research Group - University of Amsterdam, 2009.

[PRG0910] J.A. Bergstra and C.A. Middelburg, *Indirect Jumps Improve Instruction Sequence Performance,* Programming Research Group - University of Amsterdam, 2009.

[PRG0909] J.A. Bergstra and C.A. Middelburg, *Arithmetical Meadows,* Programming Research Group - University of Amsterdam, 2009.

[PRG0908] B. Diertens, *Software Engineering with Process Algebra: Modelling Client / Server Architecures,* Programming Research Group - University of Amsterdam, 2009.

[PRG0907] J.A. Bergstra and C.A. Middelburg, *Inversive Meadows and Divisive Meadows,* Programming Research Group - University of Amsterdam, 2009.

[PRG0906] J.A. Bergstra and C.A. Middelburg, *Instruction Sequence Notations with Probabilistic Instructions,* Programming Research Group - University of Amsterdam, 2009.

[PRG0905] J.A. Bergstra and C.A. Middelburg, *A Protocol for Instruction Stream Processing,* Programming Research Group - University of Amsterdam, 2009.

[PRG0904] J.A. Bergstra and C.A. Middelburg, *A Process Calculus with Finitary Comprehended Terms,* Programming Research Group - University of Amsterdam, 2009.

[PRG0903] J.A. Bergstra and C.A. Middelburg, *Transmission Protocols for Instruction Streams,* Programming Research Group - University of Amsterdam, 2009.

[PRG0902] J.A. Bergstra and C.A. Middelburg, *Meadow Enriched ACP Process Algebras,* Programming Research Group - University of Amsterdam, 2009.

[PRG0901] J.A. Bergstra and C.A. Middelburg, *Timed Tuplix Calculus and the Wesseling and van den Berg Equation,* Programming Research Group - University of Amsterdam, 2009.

[PRG0814] J.A. Bergstra and C.A. Middelburg, *Instruction Sequences for the Production of Processes,* Programming Research Group - University of Amsterdam, 2008.

[PRG0813] J.A. Bergstra and C.A. Middelburg, *On the Expressiveness of Single-Pass Instruction Sequences,* Programming Research Group - University of Amsterdam, 2008.

[PRG0812] J.A. Bergstra and C.A. Middelburg, *Instruction Sequences and Non-uniform Complexity Theory,* Programming Research Group - University of Amsterdam, 2008.

[PRG0811] D. Staudt, *A Case Study in Software Engineering with PSF: A Domotics Application,* Programming Research Group - University of Amsterdam, 2008.

[PRG0810] J.A. Bergstra and C.A. Middelburg, *Thread Algebra for Poly-Threading,* Programming Research Group - University of Amsterdam, 2008.

[PRG0809] J.A. Bergstra and C.A. Middelburg, *Data Linkage Dynamics with Shedding,* Programming Research Group - University of Amsterdam, 2008.

[PRG0808] B. Diertens, *A Process Algebra Software Engineering Environment,* Programming Research Group - University of Amsterdam, 2008.

[PRG0807] J.A. Bergstra, S. Nolst Trenite, and M.B. van der Zwaag, *Tuplix Calculus Specifications of Financial Transfer Networks,* Programming Research Group - University of Amsterdam, 2008.

[PRG0806] J.A. Bergstra and C.A. Middelburg, *Data Linkage Algebra, Data Linkage Dynamics, and Priority Rewriting,* Programming Research Group - University of Amsterdam, 2008.

[PRG0805] J.A. Bergstra, S. Nolst Trenite, and M.B. van der Zwaag, *UvA Budget Allocatie Model,* Programming Research Group - University of Amsterdam, 2008.

[PRG0804] J.A. Bergstra and C.A. Middelburg, *Thread Algebra for Sequential Poly-Threading,* Programming Research Group - University of Amsterdam, 2008.

[PRG0803] J.A. Bergstra and C.A. Middelburg, *Thread Extraction for Polyadic Instruction Sequences,* Programming Research Group - University of Amsterdam, 2008.

[PRG0802] A. Barros and T. Hou, *A Constructive Version of AIP Revisited,* Programming Research Group - University of Amsterdam, 2008.

[PRG0801] J.A. Bergstra and C.A. Middelburg, *Programming an Interpreter Using Molecular Dynamics,* Programming Research Group - University of Amsterdam, 2008.

The above reports and more are available through the website: www.science.uva.nl/research/prog/

Electronic Report Series

section Theory of Computer Science
Faculty of Science
University of Amsterdam

Science Park 904
1098 XG Amsterdam
the Netherlands

www.science.uva.nl/research/prog/