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Equations for formally real meadows

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Abstract

We consider the signatures $\Sigma_m = (0, 1, -, +, \cdot, {}^{-1})$ of meadows and $(\Sigma_m, \mathbf{s})$ of signed meadows. We give two complete axiomatizations of the equational theories of the real numbers with respect to these signatures. In the first case, we extend the axiomatization of zero-totalized fields by a single axiom scheme expressing formal realness; the second axiomatization presupposes an ordering. We apply these completeness results in order to obtain complete axiomatizations of the complex numbers.

Keywords and phrases: formally real meadow, signed meadow, real numbers, complex numbers, completeness theorems.

1 Introduction

The signature $\Sigma_f = (0, 1, -, +, \cdot)$ of *fields* has two constants 0 and 1, a unary function $-$, and two binary functions $+$ and $\cdot$. The first-order theory of fields is given by the axioms of commutative rings (see Table 1) and two additional axioms, namely

$$\begin{aligned} 0 &\neq 1, \\ x \neq 0 &\rightarrow \exists y\ x \cdot y = 1. \end{aligned}$$

A field F is said to be *ordered* if there exists a subset $F^{>0} \subseteq F$—the set of positive elements in F—such that $F^{>0}$ is closed under addition and multiplication, and F is the disjoint union of $F^{>0}$, $\{0\}$, and $\{-a \mid a \in F^{>0}\}$. Then F is totally ordered if we define $a > b$ to mean $a - b \in F^{>0}$. Moreover, if $a > b$, then $a + c > b + c$ for every c and $a \cdot c > b \cdot c$ for every $c \in F^{>0}$. The theory of ordered fields is formulated over the signature $\Sigma_{of} = (0, 1, -, +, \cdot, <)$. It has all the field axioms and, in addition, the axioms for a total ordering that is compatible with the field operations given in Table 2.

In 1927, the theory of ordered fields grew into the Artin-Schreier theory of ordered fields and formally real fields.

Definition 1.1. *A field F is called* formally real *if -1 is not a sum of squares in F.*

$$
\begin{aligned}
(x+y)+z &= x+(y+z)\\
x+y &= y+x\\
x+0 &= x\\
x+(-x) &= 0\\
(x\cdot y)\cdot z &= x\cdot(y\cdot z)\\
x\cdot y &= y\cdot x\\
1\cdot x &= x\\
x\cdot(y+z) &= x\cdot y+x\cdot z
\end{aligned}
$$

Table 1: The set CR of axioms for commutative rings

A main result of the Artin-Schreier theory (see e.g. [9]) states:

Proposition 1.2. *Let F be an arbitrary field. F is formally real if and only if for all $n \geq 0$ and all $x_0, \ldots, x_n \in F$ we have*

$$\Sigma_{i=0}^{n} x_i^2 = 0 \Rightarrow x_0 = \cdots = x_n = 0.$$

Formally real fields can therefore be axiomatized by the following infinite list of axioms, one for each $n \geq 0$,

$$\forall x_0 \forall x_1 \cdots \forall x_n (x_0 \cdot x_0 + \cdots + x_n \cdot x_n = 0 \rightarrow (x_0 = 0 \ \wedge \ \cdots \ \wedge \ x_n = 0)).$$

A formally real field has no defined order relation. However, it is always possible to find an ordering (and often more) that will change a formally real field into an ordered field. One can view a formally real field as an ordered field where the ordering is not explicitly given. The fields of rational numbers $\mathbb{Q}$ and of real numbers $\mathbb{R}$ are examples.

$$x \neq 0 \rightarrow (x < 0 \ \vee \ 0 < x) \tag{OF1}$$
$$x < y \rightarrow \neg(y < x \ \vee \ x = y) \tag{OF2}$$
$$x < y \rightarrow x + z < y + z \tag{OF3}$$
$$x < y \ \wedge \ 0 < z \rightarrow x \cdot z < y \cdot z \tag{OF4}$$

Table 2: The set OF of axioms for ordered fields

Since the signature of fields does not include a multiplicative inverse, the axiom for the inverse is not universal, and therefore a substructure of a field closed under multiplication is not always a field. This can be remedied by adding a unary inverse operation $^{-1}$ to the language. In [6] *meadows* were defined as members of a variety specified by equations. A meadow is a commutative ring equipped with a total unary operation $^{-1}$ named *inverse* that satisfies $0^{-1} = 0$. Every field F can be expanded to a meadow (or *zero-totalized field*) F_0

after making the inverse operator total by $0^{-1} = 0$. Thus $\mathbb{Q}_0$, $\mathbb{R}_0$ and $\mathbb{C}_0$ are meadows—the meadows of the rational, real and complex numbers, respectively.

An advantage of meadows over working with the signature of fields is that it facilitates formal reasoning without requiring the use of either a logic of partial operations or a three valued logic. We will exploit this advantage and prove two completeness results the statement and meaning of which are accessible to all people who have been exposed to real numbers in elementary mathematics. We will view the real numbers as a formally real meadow which can be equipped with an ordering that is encoded in a sign function. We will prove that all valid equations over the meadow of reals are derivable from the axioms of meadows plus an axiom scheme expressing formal realness; the valid equations over the signed meadow of the reals follow from the axioms of meadows together with the axioms for the sign function.

The remainder of this paper is organized as follows. The next section comprises preliminaries and a digression on meadows of characteristic 0. In Section 3 we introduce formally real meadows and provide two axiomatizations with accompanying completeness results. Our first completeness result is

$$Md + EFR \vdash s = t \quad \text{if and only if} \quad \mathbb{R}_0 \models s = t \tag{$\dagger$}$$

where *Md* is a finite equational axiomatization of meadows and *EFR* is an equational axiom scheme expressing formal realness. We obtain (†) as an immediate consequence of the Artin-Schreier Theorem and Tarski's theorem on quantifier elimination for real closed fields. Moreover, we introduce signed meadows and give a finite axiomatization of formally real meadows expanded by a sign function. Our second completeness result with respect to formally real meadows is

$$Md + Signs \vdash s = t \quad \text{if and only if} \quad (\mathbb{R}_0, \mathbf{s}) \models s = t \tag{$\ddagger$}$$

with *Signs* a finite set of axioms for the sign function $\mathbf{s}$. Also (‡) relies on both the Artin-Schreier Theorem and Tarski's theorem. In the last section, we apply these completeness results and obtain complete axiomatizations of the meadow of complex numbers.

2 Preliminaries

In [6] *meadows* were defined as the members of a variety specified by twelve equations. However, in [3] it was established that the ten equations in Table 3 imply those used in [6]. Summarizing, a meadow is a commutative ring with unit equipped with a total unary operation $^{-1}$ named *inverse* that satisfies the two equations

$$\begin{aligned} (x^{-1})^{-1} &= x, \\ x \cdot (x \cdot x^{-1}) &= x. \end{aligned} \tag{RIL}$$

Here *RIL* abbreviates *Restricted Inverse Law*. We write *Md* for the set of axioms in Table 3 and write $\Sigma_m = (\Sigma_f, ^{-1})$ for the signature of meadows. From the axioms in *Md* the following

$$
\begin{aligned}
(x+y)+z &= x+(y+z)\\
x+y &= y+x\\
x+0 &= x\\
x+(-x) &= 0\\
(x\cdot y)\cdot z &= x\cdot(y\cdot z)\\
x\cdot y &= y\cdot x\\
1\cdot x &= x\\
x\cdot(y+z) &= x\cdot y+x\cdot z\\
(x^{-1})^{-1} &= x\\
x\cdot(x\cdot x^{-1}) &= x
\end{aligned}
$$

Table 3: The set *Md* of axioms for meadows

identities are derivable:

$$
\begin{aligned}
0^{-1} &= 0,\\
(-x)^{-1} &= -(x^{-1}),\\
(x\cdot y)^{-1} &= x^{-1}\cdot y^{-1},\\
0\cdot x &= 0,\\
x\cdot -y &= -(x\cdot y),\\
-(-x) &= x.
\end{aligned}
$$

We often use the derived operators *subtraction*, *pseudo ones* and *pseudo zeros* given in Table 4. Pseudo constants enjoy a couple of nice properties which are listed in the appendix. The most prominent are $0_0 = 1_1 = 1$, $0_1 = 1_0 = 0$ and

$$0_t + 1_t = 1 \text{ and } 0_t \cdot 1_t = 0.$$

for all terms t. In the remainder we shall tacitly assume that a meadow has subtraction and pseudo constants. Moreover, we freely use numerals $\underline{n}$—defined by $\underline{0} = 0$, $\underline{1} = 1$ and $\underline{n+1} = \underline{n} + 1$ for $n \geq 1$—and exponentiation with constant integer exponents.

$$
\begin{aligned}
x-y &= x+(-y)\\
1_x &= x\cdot x^{-1}\\
0_x &= 1-1_x
\end{aligned}
$$

Table 4: The derived operators subtraction, pseudo ones and pseudo zeros

The term *cancellation meadow* is introduced in [4] for a zero-totalized field that satisfies the so-called "cancellation axiom"

$$x \neq 0 \ \wedge \ x \cdot y = x \cdot z \ \rightarrow \ y = z.$$

An equivalent version of the cancellation axiom that we shall further use in this paper is the *Inverse Law* (*IL*), i.e., the conditional axiom

$$x \neq 0 \ \rightarrow \ x \cdot x^{-1} = 1. \tag{IL}$$

So *IL* states that there are no proper zero divisors. (Another equivalent formulation of the cancellation property is $x \cdot y = 0 \ \rightarrow \ x = 0 \ \vee \ y = 0$.) Paradigm cancellation meadows are $\mathbb{Q}_0$, $\mathbb{R}_0$ and $\mathbb{C}_0$. However, there also exist meadows with proper zero devisors and infinite meadows with characteristic 0 different from $\mathbb{Q}_0$, $\mathbb{R}_0$ and $\mathbb{C}_0$. For example, in [7] it is proved that $\mathbb{Z}/n\mathbb{Z}$ with elements $\{0, 1, \ldots, n-1\}$ where arithmetic is performed modulo n is a meadow if n is squarefree, i.e. n is the product of pairwise distinct primes. Thus $\mathbb{Z}/10\mathbb{Z}$ is a meadow where $2 \neq 0 \neq 5$ but $2 \cdot 5 = 0$. The existence of an infinite non-cancellation meadow is shown in the following theorem.

Theorem 2.1. *There exists a non-cancellation meadow M of characteristic 0 which does not have $\mathbb{Q}_0$ as a subalgebra.*

Proof. Choose a new constant symbol a. For $k \in \mathbb{N}$ let

$$E_k = \{a \neq 0\} \cup \{\underline{n} \neq 0 \mid n \in \mathbb{N}, 0 < n < k\} \cup \{\underline{2} \cdot a = 0\} \cup Md.$$

Moreover, choose a prime $p \neq 2$ exceeding k and interpret a in $\mathbb{Z}/2p\mathbb{Z}$ by p. Then $\mathbb{Z}/2p\mathbb{Z} \models E_k$. It follows that E_k is consistent and therefore $E = \bigcup_{k=1}^{\infty} E_k$ is consistent by the compactness theorem. Let M be a model for E. Then

1. M is a meadow, since $Md \subseteq E$,
2. M has characteristic 0, since $M \models \underline{n} \neq 0$ for all $n \in \mathbb{N}$ with $n \neq 0$,
3. M is not a cancellation meadow, since $M \models \underline{2} \neq 0$, $M \models a \neq 0$, but $M \models \underline{2} \cdot a = 0$, and
4. $M \models \underline{2} \cdot \underline{2}^{-1} \neq 1$, for otherwise

$$a = 1 \cdot a = \underline{2} \cdot \underline{2}^{-1} \cdot a = \underline{2}^{-1} \cdot \underline{2} \cdot a = \underline{2}^{-1} \cdot 0 = 0;$$

hence $\mathbb{Q}_0$ is not a subalgebra of M.

□

In [2], we proved a finite basis result for the equational theory of cancellation meadows. This result is formulated in a generic way so that it can be applied to any expansion of a meadow that satisfies the propagation properties defined below.

Definition 2.2. *Let Σ be an extension of Σ_m and $E \supseteq Md$.*

1. (Σ, E) *has the* propagation property for pseudo ones *if for each pair of Σ-terms t, r and context $C[\ ]$,*
$$E \vdash 1_t \cdot C[r] = 1_t \cdot C[1_t \cdot r].$$

2. (Σ, E) *has the* propagation property for pseudo zeros *if for each pair of Σ-terms t, r and context $C[\ ]$,*
$$E \vdash 0_t \cdot C[r] = 0_t \cdot C[0_t \cdot r].$$

Preservation of these propagation properties admits the following nice result:

Theorem 2.3 (Generic Basis Theorem for Cancellation Meadows)**.** *If $\Sigma \supseteq \Sigma_m$, $E \supseteq Md$ and (Σ, E) has the pseudo one propagation property and the pseudo zero propagation property, then*

$$E \vdash s = t \quad \text{if and only if} \quad E + IL \models s = t$$

for all $s, t \in \Sigma$.

Proof. This is Theorem 3.1 of [2]. □

Meadow terms can be represented in a particular standard way.

Definition 2.4. *A term P over Σ_m is a* Standard Meadow Form (SMF) *if, for some $n \in \mathbb{N}$, P is an* SMF of level n*. SMFs of level n are defined as follows:*

SMF of level 0*: each expression of the form $s \cdot t^{-1}$ with s and t Σ_f-terms,*

SMF of level $n+1$: each expression of the form

$$0_t \cdot P + 1_t \cdot Q$$

with t a Σ_f-term and P and Q SMFs of level n.

Theorem 2.5. *For each term s over Σ_m there exists an SMF s' with the same variables such that $Md \vdash s = s'$.*

Proof. This is Theorem 2.1 of [2]. □

It follows that every meadow equation has a first-order representation over the signature of fields. Since we will apply the first-order representation solely in the context of cancellation meadows we may freely use *IL*.

Corollary 2.6. *For each equation $s = t$ over Σ_m there exists a quantifier-free first-order formula $\phi(s, t)$ over Σ_f with the same variables such that*

$$Md + IL \vdash s = t \leftrightarrow \phi(s, t).$$

Proof. By the preceding theorem we may assume that both s and t are SMF's. We employ induction on the levels n, m of s and t, respectively.

$1_{\underline{n+1}} = 1$	$(\mathrm{C0}_n)$

Table 5: The set of axioms *C0* for meadows of characteristic 0

1. $n = 0$: Then there are terms s_1, s_2 over Σ_f such that $s = s_1 \cdot s_2^{-1}$.

   (a) $m = 0$: Then there are terms t_1, t_2 over Σ_f such that $t = t_1 \cdot t_2^{-1}$. Since $s = t$ we have $s_2 \cdot t_2 \cdot (s_1 \cdot t_2 - t_1 \cdot s_2) = 0$ and hence $s_2 = 0$ or $t_2 = 0$ or $s_1 \cdot t_2 = t_1 \cdot s_2$. Thus we can take

$$\begin{aligned} \phi(s,t) \quad \equiv \quad & (s_2 = 0 \to (t_1 = 0 \ \vee \ t_2 = 0)) \ \wedge \\ & (t_2 = 0 \to (s_1 = 0 \ \vee \ s_2 = 0)) \ \wedge \\ & (s_2 \cdot t_2 \neq 0 \to s_1 \cdot t_2 = t_1 \cdot s_2). \end{aligned}$$

   (b) $m = k + 1$: Then $t = 0_{t'} \cdot P + 1_{t'} \cdot Q$ for some Σ_f term t' and SMFs of level k. Observe that if $t' = 0$, then $1_{t'} = 0$ and $0_{t'} = 1$ and therefore $s = P$. Likewise, if $t' \neq 0$, then $1_{t'} = 1$ and $0_{t'} = 0$ and therefore $s = Q$. Thus we can take

$$\phi(s,t) \equiv (t' = 0 \to \phi(s,P)) \ \wedge \ (t' \neq 0 \to \phi(s,Q)).$$

2. $n = l + 1$: Then $s = 0_{s'} \cdot P + 1_{s'} \cdot Q$ for some Σ_f-term s' and SMFs P, Q of level k. Now we argue as in case 1(b) and take

$$\phi(s,t) \equiv (s' = 0 \to \phi(P,t)) \ \wedge \ (s' \neq 0 \to \phi(Q,t)).$$

□

We use pseudo ones to give an infinite axiomatization of meadows of characteristic 0 in Table 5. As a corollary to Theorem 2.3 we have:

Corollary 2.7. *Let $s, t \in \Sigma_m$. Then*

$$Md + C0 \vdash s = t \quad \textit{if and only if} \quad Md + C0 + IL \models s = t.$$

Proof. Since (Md, Σ_m) has the propagation properties (see Corollary 3.1 of [2]), the derived operators also share these properties. We therefore can apply Theorem 2.3. □

An *extension field* $\widehat{F}$ of a field F is any field containing F as a subalgebra. An element x of $\widehat{F}$ is said to be *algebraic over* F if there exists a non-zero polynomial P with coefficients from F such that $P(x) = 0$. An extension field $\widehat{F}$ of F is said to be *algebraically closed* if all its elements are algebraic. Every field has an algebraic closure. In 1951, Tarski [12] proved that the theory of algebraically closed fields in the first-order language over the signature $(0, 1, -, +, \cdot)$ admits elimination of quantifiers. The most important model theoretic consequences hereof are the completeness of the theory of algebraically closed fields of a given characteristic—in

$0_{x_0^2+\cdots+x_n^2} \cdot x_0 = 0$	(EFR$_n$)

Table 6: The set *EFR* of axioms for formally real meadows

particular, the theory of algebraically closed fields of characteristic 0 coincides with the theory of the complex numbers over the signature $(0, 1, -, +, \cdot)$—and its decidability (see e.g. [8, 10]).

A reduct of an algebraic structure is obtained by omitting some of the operations and relations of that structure. The converse of a reduct is an expansion. In the sequel, we will write $M|\Sigma'$ for the reduct of the Σ-algebra M to $\Sigma' \subseteq \Sigma$ and $(M, \diamond)$ for the expansion of M by an operation or relation $\diamond$. In particular, we will write $M|\Sigma_f$ for the reduct of the meadow M to the signature of fields, and $(M, \mathbf{s})$ and $(M, <)$ for the expansion of the meadow M with a sign function and order relation, respectively.

Theorem 2.8. *Let s, t be Σ_m-terms. Then $Md + C0 \vdash s = t$ if and only if $\mathbb{C}_0 \models s = t$.*

Proof. Because $\mathbb{C}_0$ constitutes a meadow of characteristic 0, soundness is immediate.

Assume that $s = t$ is not derivable from $Md + C0$. By the previous corollary, there exists a cancellation meadow M of characteristic 0 with $M \not\models s = t$. Since M is a field, it has an algebraic closure, say, $\widehat{M}$. Then $\widehat{M} \not\models s = t$ and hence $\widehat{M}|\Sigma_f \not\models \phi(s, t)$ where $\phi(s, t)$ is the Σ_f-representation given in Corollary 2.6. Therefore $\mathbb{C}_0|\Sigma_f \not\models \phi(s, t)$ by completeness and thus $\mathbb{C}_0 \not\models s = t$. □

Remark 2.9. *Initial algebras provide standard models of equational specifications. They contain only elements that can be constructed from those appearing in the specification, and satisfy only those closed equations that appear in the specification or are logical consequences of them. It is easy to see that Md+ C0 constitutes an initial algebra specification of the rational numbers. See also [6].*

Remark 2.10. *In a way very similar to Theorem 2.8 one can show that $Md + \{\underline{p} = 0\}$ is a complete axiomatization of algebraically closed fields of characteristic p.*

3 Formally real meadows

Our first axiomatization of formal realness is the infinite set of axioms given in Table 6.

Definition 3.1. *A meadow M is called formally real if $M \models EFR$.*

Observe that (EFR$_n$) is an equational representation of the conditional axiom

$$CEFR_n \equiv x_0^2 + \cdots x_n^2 = 0 \rightarrow x_0 = 0.$$

Proposition 3.2. *For all $n \geq 0$, $Md + EFR \vdash CEFR_n$.*

Proof. Suppose $x_0^2 + \cdots x_n^2 = 0$. Then $0_{x_0^2+\cdots x_n^2} = 0_0 = 1$. Hence $x_0 = 0$ by (EFR$_n$). □

Again $\mathbb{Q}_0$ and $\mathbb{R}_0$ are formally real meadows; however, $\mathbb{C}_0$ is not formally real.

As a corollary to Theorem 2.3 we have

Corollary 3.3. *Let $s, t \in \Sigma_m$. Then*

$$Md + EFR \vdash s = t \quad \textit{if and only if} \quad Md + EFR + IL \vdash s = t.$$

Proof. This follows again from the fact that (Md, Σ_m) has the propagation properties (see Corollary 3.1 of [2]). □

Remark 3.4. *As an alternative we could have chosen the axiom scheme AEFR given in Table 7 as an axiomatization of formal realness. Observe that* (AEFR$_n$) *expresses the fact that*

$$\Sigma_{i=0}^{n} x_i^2 \neq -1$$

which is somewhat closer to the definition of formal realness found in the literature (see also Definition 1.1). In the following theorem we prove that EFR and AEFR are indeed equivalent in the context of meadows.

$1_{1+x_0^2+\cdots+x_n^2} = 1$	(AEFR$_n$)

Table 7: The set *AEFR* of alternative axioms for formally real meadows

Theorem 3.5. *Let M be a meadow and $s, t \in \Sigma_m$. Then*

$$Md + EFR \vdash s = t \quad \textit{if and only if} \quad Md + AEFR \vdash s = t.$$

Proof. Observe that Corollary 3.3 also holds for *AEFR* instead of *EFR*. It therefore suffices to prove that

$$M \models EFR \quad \text{if and only if} \quad M \models AEFR$$

for every cancellation meadow M. The claim now follows from Proposition 1.2 and the fact that a cancellation meadow is a field. □

A *real closed* field F is a formally real field in which every positive element has a square root and, moreover, every polynomial of odd degree in one indeterminate with coefficients in F has a root. In [12], Tarski also proved quantifier elimination for real closed fields in the first-order language over the signature $(0, 1, -, +, \cdot, <)$. As a consequence, the theory of real closed fields is complete —in particular, the theory of real closed fields coincides with the theory of the real numbers over the signature $(0, 1, +, \cdot, -, <)$—and is decidable (see e.g. [8, 10]).

If $(F, <)$ is an ordered field, the *Artin-Schreier Theorem* [1] states that F has an algebraic extension, called the *real closure* $\widehat{F}$ of F, such that $\widehat{F}$ is a real closed field whose ordering is an extension of the given ordering $<$ on F and is unique up to isomorphism of fields. The classical proof of this theorem relies on the use of Zorn's Lemma which is equivalent to the

axiom of choice. However, in 1991 Sander [11] showed the existence of real closures of ordered fields in Zermelo-Fränkel axiomatic set theory.

Our first completeness result for formally real meadows follows the proof of Theorem 2.8 and, in addition, relies on the Artin-Schreier Theorem and Tarski's theorem for real closed fields.

Theorem 3.6. *Let s, t be Σ_m-terms. Then $Md + EFR \vdash s = t$ if and only if $\mathbb{R}_0 \models s = t$.*

Proof. Because the meadow of real numbers is formally real, soundness is immediate.

Assume that $s = t$ is not derivable from $Md{+}EFR$. From Corollary 3.3 we get the existence of a formally real cancellation meadow M with $M \not\models s = t$. Since M is formally real, it is orderable by, say, $<$. By the Artin-Schreier Theorem $(M, <)$ has the real closure $(\widehat{M}, <)$. Since $(\widehat{M}, <) \not\models s = t$, we find that $(\widehat{M}, <)|\Sigma_f \not\models \phi(s, t)$ where $\phi(s, t)$ is the first-order representation of $s = t$ given in Corollary 2.6. Since the theory of real closed fields coincides with the theory of the real numbers over the signature $(0, 1, -, +, \cdot, <)$, we can conclude that $\mathbb{R}_0|\Sigma_f \not\models \phi(s, t)$ and therefore $\mathbb{R}_0 \not\models s = t$. □

A finite axiomatization of formal realness is obtained by extending the signature Σ_m of meadows with the unary sign (or signum) function $\mathbf{s}(x)$. We write Σ_{fs}, Σ_{ms} for these extended signatures, so $\Sigma_{fs} = (\Sigma_f, \mathbf{s})$ and $\Sigma_{ms} = (\Sigma_m, \mathbf{s})$. The sign function $\mathbf{s}(x)$ presupposes an ordering on its domain and is defined by

$$\mathbf{s}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

We define the sign function in an equational manner by the set *Signs* of axioms given in Table 8. First, notice that by *Md* and axiom (S1) (or axiom (S2)) we find

$$\mathbf{s}(0) = 0 \quad \text{and} \quad \mathbf{s}(1) = 1.$$

Then, observe that in combination with the inverse law *IL*, axiom (S6) is an equational representation of the conditional axiom

$$\mathbf{s}(x) = \mathbf{s}(y) \;\rightarrow\; \mathbf{s}(x + y) = \mathbf{s}(x).$$

From *Md* and axioms (S3)–(S6) one can easily compute $\mathbf{s}(t)$ for any closed term t. Some more consequences of the $Md + Signs$ axioms are these (see [2]):

$$\mathbf{s}(x^2) = 1_x \tag{S7}$$
$$\mathbf{s}(x^3) = \mathbf{s}(x) \tag{S8}$$
$$1_x \cdot \mathbf{s}(x) = \mathbf{s}(x) \tag{S9}$$
$$\mathbf{s}(x)^{-1} = \mathbf{s}(x) \tag{S10}$$
$$\mathbf{s}(\mathbf{s}(x)) = \mathbf{s}(x) \tag{S11}$$

$\mathbf{s}(1_x) = 1_x$	(S1)
$\mathbf{s}(0_x) = 0_x$	(S2)
$\mathbf{s}(-1) = -1$	(S3)
$\mathbf{s}(x^{-1}) = \mathbf{s}(x)$	(S4)
$\mathbf{s}(x \cdot y) = \mathbf{s}(x) \cdot \mathbf{s}(y)$	(S5)
$0_{\mathbf{s}(x)-\mathbf{s}(y)} \cdot (\mathbf{s}(x+y) - \mathbf{s}(x)) = 0$	(S6)

Table 8: The set *Signs* of axioms for the sign function

Expanding $\mathbb{Q}_0$ and $\mathbb{R}_0$ in the obvious way, we obtain signed meadows $(\mathbb{Q}_0, \mathbf{s})$ and $(\mathbb{R}_0, \mathbf{s})$.

Signed meadows need not be cancellation meadows. This can easily be seen as follows. Expand the signature of signed meadows by two new constants a, b and consider the equational theory $T = Md + Signs + \{a \cdot b = 0\}$. We obtain a model for T by interpreting in $(\mathbb{R}_0, \mathbf{s})$ a by 0 and b by 1 and construct a second model by interchanging the interpretations of a and b. It follows that the initial algebra of T satisfies neither $a = 0$ nor $b = 0$. However, $Md \cup Signs$ has the propagation properties for pseudo ones and zeros (see Corollary 5.1 of [2]). We therefore have the following finite basis result for signed cancellation meadows.

Corollary 3.7. *Let $s, t \in \Sigma_{ms}$. Then*

$$Md + Signs \vdash s = t \quad \text{if and only if} \quad Md + Signs + IL \models s = t.$$

Signed meadows are formally real.

Proposition 3.8. *For every $n \geq 0$, $Md + Signs \vdash EFR_n$.*

Proof. We give a semantic proof; a syntactic proof can be found in the appendix. By Corollary 3.7 it suffices to prove that $M \models EFR_n$ for all signed cancellation meadows. Thus assume $M \models Md + Signs + IL$. Observe that if $a_0^2 + \cdots + a_n^2 \neq 0$ then $1_{a_0^2+\cdots+a_n^2} = 1$ and hence $0_{a_0^2+\cdots+a_n^2} \cdot a_0 = 0$. We can therefore assume that $a_0^2 + \cdots + a_n^2 = 0$ and prove $M \models EFR_n$ by induction on n. EFR_0 follows from the cancellation property. For $n = m + 1$ we distinguish three cases.

Case $a_0 = 0$: Then $0_{a_0^2+\cdots+a_{m+1}^2} \cdot a_0 = 0$.

Case $a_i = 0$ for some $0 < i \leq m + 1$: Without loss of generality we may assume that $i = m + 1$. Then $a_0^2 + \cdots + a_{m+1}^2 = a_0^2 + \cdots + a_m^2$. Hence

$$0_{a_0^2+\cdots+a_{m+1}^2} \cdot a_0 = 0_{a_0^2+\cdots+a_m^2} \cdot a_0 = 0$$

by the induction hypothesis.

Case $a_i \neq 0$ for all $0 \leq i \leq m + 1$: We obtain a contradiction as follows. Since $a_0^2 + \cdots + a_{m+1}^2 = 0$ we have $\mathbf{s}(a_0^2 + \cdots + a_{m+1}^2) = \mathbf{s}(0) = 0$. On the other hand, it follows from the cancellation property and (S7) that $\mathbf{s}(a_i^2) = 1$ for every $0 \leq i \leq m + 1$ and hence $\mathbf{s}(a_0^2 + \cdots + a_{m+1}^2) = 1$ by (S6). □

We can represent signed meadow terms in a standard way similar to ordinary meadow terms.

Definition 3.9. *A term P over Σ_{ms} is a* Signed Standard Meadow Form (SSMF) *if, for some $n \in \mathbb{N}$, P is an* SSMF of level n. *SSMFs of level n are defined as follows:*

SSMF of level 0*: each expression of the form $s \cdot t^{-1}$ with s and t Σ_{fs}-terms,*

SSMF of level $n+1$: each expression of the form

$$0_t \cdot P + 1_t \cdot Q$$

with t a Σ_{fs}-term and P and Q SSMFs of level n.

Theorem 3.10. *For each term s over Σ_{ms} there exists an SSMF s' with the same variables such that $Md + Signs \vdash s = s'$.*

Proof. In [2] a proof is given for Σ_m-terms by structural induction. It thus suffices to prove that the set of SSMFs is closed under $\mathbf{s}$. We employ induction on level height n. If the SSMF t has level 0 then $t = s_1 \cdot {s_2}^{-1}$ for Σ_{fs}-terms s_1, s_2. Then

$$\mathbf{s}(t) = \mathbf{s}(s_1 \cdot s_2^{-1}) = \mathbf{s}(s_1) \cdot \mathbf{s}(s_2^{-1}) = \mathbf{s}(s_1) \cdot \mathbf{s}(s_2) = \mathbf{s}(s_1 \cdot s_2) \cdot 1^{-1}$$

by axiom (S4) and (S5). Assume $t = 0_s \cdot P + 1_s \cdot Q$ for some Σ_{fs}-term s and SSMFs P, Q of level n. Then $0_s \cdot t = 0_s \cdot P$ and $1_s \cdot t = 1_s \cdot Q$. Hence

$$\begin{aligned}
\mathbf{s}(t) &= (0_s + 1_s) \cdot \mathbf{s}(t) \\
&= 0_s \cdot \mathbf{s}(t) + 1_s \cdot \mathbf{s}(t) \\
&= \mathbf{s}(0_s \cdot t) + \mathbf{s}(1_s \cdot t) \\
&= 0_s \cdot \mathbf{s}(P) + 1_s \cdot \mathbf{s}(Q)
\end{aligned}$$

by (S1), (S2) and (S5). □

As in the case of ordinary meadow equations, it follows that every signed meadow equation has a first-order representation over the signature of signed fields. In the corollary and the proposition below we again apply freely *IL* since we will use these results only in the context of cancellation meadows.

Corollary 3.11. *For each equation $s = t$ over Σ_{ms} there exists a quantifier-free first-order formula $\phi(s,t)$ over Σ_{fs} with the same variables such that*

$$Md + Signs + IL \vdash s = t \leftrightarrow \phi(s,t).$$

Given a signed cancellation meadow, we shall consider it ordered by the order induced by the sign function. This can be expressed by the axiom

$$x < y \leftrightarrow \mathbf{s}(y - x) = 1. \qquad (SO)$$

Proposition 3.12.

1. *$Md + IL + Signs + SO \vdash OF$, and*
2. *for all terms s, t over Σ_{fs} there exists a formula $\psi(s,t)$ over $(\Sigma_f, <)$ with the same free variables such that*
$$Md + IL + Signs + SO \vdash s = t \leftrightarrow \psi(s,t).$$

Proof. 1. We prove the derivability of the axioms given in Table 2.

(OF1): Assume $x \neq 0$. Then $1_x = 1$. Therefore

$$\begin{aligned}(1 - \mathbf{s}(x)) \cdot (1 + \mathbf{s}(x)) &= 1_x \cdot (1 - \mathbf{s}(x)^2) \\ &= 1_x - \mathbf{s}(x)^2 && \text{by (S9)} \\ &= 1_x - 1_x && \text{by (S5), (S7)} \\ &= 0.\end{aligned}$$

Hence $\mathbf{s}(x) = 1$ or $\mathbf{s}(x) = -1$ by cancellation. If $\mathbf{s}(x) = 1$ then $0 < x$; if $\mathbf{s}(x) = -1$ then $1 = -1 \cdot \mathbf{s}(x) = \mathbf{s}(-1) \cdot \mathbf{s}(x) = \mathbf{s}(-x)$ and thus $x < 0$.

(OF2): Assume $x < y$. Then $\mathbf{s}(y - x) = 1$. Thus, if $y < x$ then $\mathbf{s}(x - y) = \mathbf{s}(y - x)$ and so $0 = \mathbf{s}(0) = \mathbf{s}((y - x) + (x - y)) = 1$ by (S6). Likewise, if $x = y$, then $0 = \mathbf{s}(x - x) = \mathbf{s}(y - x) = 1$. Thus $\neg(y < x \ \vee \ x = y)$.

(OF3): If $x < y$ then $1 = \mathbf{s}(y - x) = \mathbf{s}((y + z) - (x + z))$. Hence $x + z < y + z$.

(OF4): If $x < y$ and $0 < z$ then $\mathbf{s}(y - x) = 1$ and $\mathbf{s}(z) = 1$. Thus $1 = 1 \cdot 1 = \mathbf{s}(y - x) \cdot \mathbf{s}(z) = \mathbf{s}((y - x) \cdot z) = \mathbf{s}(y \cdot z - x \cdot z)$, i.e. $x \cdot z < y \cdot z$.

2. Let t be a Σ_{fs}-term, $Var(t)$ be the set of variables occurring in t and $x \notin Var(t)$. By structural induction, we will first construct a formula $\gamma(x,t)$ over $(\Sigma_f, <)$ with free variables $Var(t) \cup \{x\}$ such that for all $(\Sigma_f, \mathbf{s})$-terms s, $M' \models s = t \leftrightarrow \gamma(s,t)$.

   (a) $t = 0$, $t = 1$ or $t = y$ $(y \not\equiv x)$: $\gamma(x,t) \equiv x = t$,

   (b) $t \equiv -t'$: $\gamma(x,t) \equiv \forall z(\gamma(z,t') \to x = -z)$ with $z \notin Var(t') \cup \{x\}$,

   (c) $t \equiv t_1 \diamond t_2$ with $\diamond \in \{+, \cdot\}$: $\gamma(x,t) \equiv \forall y \forall z(\gamma(y,t_1) \ \wedge \ \gamma(z,t_2) \to x = y \diamond z)$ with $y, z \notin Var(t_1 \diamond t_2) \cup \{x\}$,

   (d) $t \equiv \mathbf{s}(t')$:

$$\begin{aligned}\gamma(x,t) \quad \equiv \quad & (x = 0 \to \gamma(x,t')) \ \wedge \\ & (x \neq 0 \to \forall z(\gamma(z,t') \to \\ & \qquad ((0 < z \ \wedge \ x = 1) \ \vee \ (z < 0 \ \wedge \ x = -1))))\end{aligned}$$

   with $z \notin Var(t') \cup \{x\}$.

We prove that $\gamma(x, \mathbf{s}(t'))$ meets the requirements. $\gamma(x, \mathbf{s}(t'))$ is a formula over $(\Sigma_f, <)$ since $\gamma(x,t')$ is so. Moreover, the free variables in $\gamma(x, \mathbf{s}(t'))$ are $Var(t') \cup \{x\} = Var(\mathbf{s}(t')) \cup \{x\}$. Now assume $s = \mathbf{s}(t')$. If $s = 0$ then $\mathbf{s}(t') = 0$. If $t' \neq 0$ we can

argue as in (1) in the case of (OF1) to derive a contradiction. Thus $t' = 0$. Hence $s = t'$ and therefore $\gamma(s, t')$. To prove the second conjunct assume $s \neq 0$ and $z = t'$. Then $z \neq 0$ and hence $0 < z$ or $z < 0$ by (OF1). If $0 < z$ then $s = \mathbf{s}(z) = 1$; likewise, if $z < 0$ then $\mathbf{s}(-z) = 1$ and hence $s = \mathbf{s}(z) = -1$. $\gamma(s, \mathbf{s}(t')) \to s = \mathbf{s}(t')$ follows in a similar fashion. Now observe that

$$M' \models s = t \leftrightarrow \forall x \forall y((x = s \ \wedge \ y = t) \to x = y)$$

with $x, y \notin Var(s) \cup Var(t)$. Thus we can take

$$\psi(s, t) \equiv \forall x \forall y((\gamma(x, s) \ \wedge \ \gamma(y, t)) \to x = y).$$

□

We slightly modify the proof of Theorem 3.6 in order to obtain:

Theorem 3.13. *Let s, t be Σ_{ms}-terms. Then Md+Signs $\vdash s = t$ if and only if $(\mathbb{R}_0, \mathbf{s}) \models s = t$.*

Proof. Soundness is again immediate.

Assume that $s = t$ is not derivable from *Md+Signs*. By Corollary 3.7 we now have a counter model M which is a signed cancellation meadow. Since M is formally real by Proposition 3.8, it is orderable. We consider the particular order given by

$$x < y \leftrightarrow \mathbf{s}(y - x) = 1.$$

Then $(M, <)$ is an ordered field by Proposition 3.12.1. By the Artin-Schreier Theorem $(M, <)$ has real closure $(\widehat{M}, <)$ whose ordering is an extension of the given ordering. We find that $(\widehat{M}, <) \not\models s = t$. By Corollary 3.11, $s = t$ is equivalent to a $(\Sigma_f, \mathbf{s})$-formula $\phi(s, t)$. Every equation $s' = t'$ in $\phi(s, t)$ can be replaced by a $(\Sigma_f, <)$-formula $\psi(s', t')$ by Proposition 3.12.2. Thus there exists a $(\Sigma_f, <)$-formula Γ such that $(\widehat{M}, <) \models s = t \leftrightarrow \Gamma$ and hence $(\widehat{M}, <)|\Sigma_f \not\models \Gamma$. Since the theory of real closed fields coincides with the theory of the real numbers over the signature $(0, 1, -, +, \cdot, <)$, we can conclude that $\mathbb{R}_0 \not\models \Gamma$. Therefore $(\mathbb{R}_0, \mathbf{s}) \not\models s = t$. □

Corollary 3.14. *If an open quantifier free formula $\phi(x_1, ..., x_k)$ is satisfiable in some signed cancellation meadow then that formula is satisfiable in the signed meadow of reals.*

Proof. This is the same fact in another wording. If $\phi(x_1, ..., x_k)$ is not satisfiable in the signed reals then $\neg\phi(x_1, ..., x_k)$ is valid in the reals and by the preceding theorem it must be derivable from *Md+ Signs*, whence it is true in all cancellation meadows in contradiction with the assumption. □

Remark 3.15. *It should be noted that whether or not the equational theory of signed rationals can be finitely axiomatized is still an open problem. The case of the reals is much simpler indeed.*

$\overline{0} = 0$	(CC0)
$\overline{1} = 1$	(CC1)
$\overline{i} = -i$	(CC2)
$\overline{-x} = -\overline{x}$	(CC3)
$\overline{x+y} = \overline{x} + \overline{y}$	(CC4)
$\overline{x \cdot y} = \overline{x} \cdot \overline{y}$	(CC5)
$\overline{x^{-1}} = \overline{x}^{-1}$	(CC6)
$\overline{\overline{x}} = x$	(CC7)
$i \cdot i = -1$	(CC8)
$1_{\overline{x}} = 1_x$	(CC9)

Table 9: The set *CC* of axioms for complex conjugation

$1_{1+x_0 \cdot \overline{x_0} + \cdots + x_n \cdot \overline{x_n}} = 1$	(SSAV$_n$)

Table 10: The set *SSAV* of axioms for the absolute value of complex numbers

4 The meadow of complex numbers

Rational complex numbers have been studied before in [5]. In this section, we generalize that approach in order to give a completeness result for arbitrary complex meadow terms.

We extend the signature Σ_m of meadows to the signature of complex numbers $(\Sigma_m, i, ^-)$ where i is the imaginary unit and $^-$ is the unary operation of complex conjugation. The defining equations are listed in Table 9 and Table 10. (CC0)—(CC8) are the usual equalities for complex numbers and (CC9) ensures propagation; *SSAV* expresses the fact that any sum of squares of absolute values of complex numbers cannot yield -1. As a special instance of *SSAV* we obtain the set of axioms *C0* for meadows of characteristic 0 (see also Table 5):

$$1_{\underline{n+1}} = 1_{1+1 \cdot \overline{1} + \cdots + 1 \cdot \overline{1}} = 1 \qquad \text{(C0}_n\text{)}$$

In the sequel we will use the derived unary operators $re(_)$ and $im(_)$—the real part and the imaginary part of a complex number—given in Table 11. Here we write $\frac{s}{t}$ for $s \cdot t^{-1}$. $re(_)$ and $im(_)$ enjoy a couple of nice properties which are listed in Table 12.

Proposition 4.1. *For* $i \in \{0, \ldots, 22\}$,

$$Md + CC + SSAV \vdash RIi$$

Proof. We prove (RI0), (RI1), (RI2), (RI13) and (RI21).

$$re(x) = \frac{1}{\underline{2}} \cdot (x + \overline{x})$$
$$im(x) = -\frac{i}{\underline{2}} \cdot (x - \overline{x})$$

Table 11: The derived operators $re(_)$ and $im(_)$

$$x = re(x) + im(x) \cdot i \tag{RI0}$$
$$re(x) = \overline{re(x)} \tag{RI1}$$
$$im(x) = \overline{im(x)} \tag{RI2}$$
$$re(re(x)) = re(x) \tag{RI3}$$
$$re(im(x)) = im(x) \tag{RI4}$$
$$im(re(x)) = 0 \tag{RI5}$$
$$im(im(x)) = 0 \tag{RI6}$$
$$re(0) = 0 \tag{RI7}$$
$$re(1) = 1 \tag{RI8}$$
$$re(i) = 0 \tag{RI9}$$
$$re(-x) = -re(x) \tag{RI10}$$
$$re(x + y) = re(x) + re(y) \tag{RI11}$$
$$re(x \cdot y) = re(x) \cdot re(y) - im(x) \cdot im(y) \tag{RI12}$$
$$re(x^{-1}) = re(x) \cdot (re(x) \cdot re(x) + im(x) \cdot im(x))^{-1} \tag{RI13}$$
$$re(\overline{x}) = re(x) \tag{RI14}$$
$$im(0) = 0 \tag{RI15}$$
$$im(1) = 0 \tag{RI16}$$
$$im(i) = 1 \tag{RI17}$$
$$im(-x) = -im(x) \tag{RI18}$$
$$im(x + y) = im(x) + im(y) \tag{RI19}$$
$$im(x \cdot y) = re(x) \cdot im(y) + im(x) \cdot re(y) \tag{RI20}$$
$$im(x^{-1}) = -im(x) \cdot (re(x) \cdot re(x) + im(x) \cdot im(x))^{-1} \tag{RI21}$$
$$im(\overline{x}) = -im(x) \tag{RI22}$$

Table 12: Properties of the real part and the imaginary part of complex numbers

- (RI0):

$$\begin{aligned} re(x) + im(x) \cdot i &= \tfrac{1}{2} \cdot (x + \overline{x}) - \tfrac{i}{2} \cdot (x - \overline{x}) \cdot i \\ &= \tfrac{1}{2} \cdot (x + \overline{x}) + \tfrac{1}{2} \cdot (x - \overline{x}) && \text{by (CC8)} \\ &= \tfrac{1}{2} \cdot x + \tfrac{1}{2} \cdot x \\ &= 2 \cdot 2^{-1} \cdot x \\ &= x && \text{by (C0}_1\text{)} \end{aligned}$$

- (RI1):

$$\begin{aligned} \overline{re(x)} &= \overline{\tfrac{1}{2}(x + \overline{x})} \\ &= \tfrac{1}{2}\overline{(x + \overline{x})} && \text{by (CC1), (CC4), (CC5), (CC6)} \\ &= \tfrac{1}{2}(\overline{x} + \overline{\overline{x}}) && \text{by (CC4)} \\ &= \tfrac{1}{2}(\overline{x} + x) && \text{by (CC7)} \\ &= \tfrac{1}{2}(x + \overline{x}) \\ &= re(x) \end{aligned}$$

- (RI2):

$$\begin{aligned} \overline{im(x)} &= \overline{-\tfrac{i}{2}(x - \overline{x})} \\ &= \tfrac{i}{2}\overline{(x - \overline{x})} && \text{by (CC1), (CC2), (CC3), (CC4), (CC5), (CC6)} \\ &= \tfrac{i}{2}(\overline{x} - \overline{\overline{x}}) && \text{by (CC3), (CC4)} \\ &= \tfrac{i}{2}(\overline{x} - x) && \text{by (CC7)} \\ &= -\tfrac{i}{2}(x - \overline{x}) \\ &= im(x) \end{aligned}$$

- (RI13), (RI21): Observe that

$$\begin{aligned} t^{-1} &= t^{-1} \cdot 1_{t^{-1}} \\ &= t^{-1} \cdot 1_t \\ &= t^{-1} \cdot 1_{\overline{t}} && \text{by (CC9)} \\ &= t^{-1} \cdot \overline{t}^{-1} \cdot \overline{t} && \text{by (CC5), (CC6)} \\ &= (t \cdot \overline{t})^{-1} \cdot \overline{t} \\ &= (re(t) \cdot re(t) + im(t) \cdot im(t))^{-1} \cdot (re(t) - im(t) \cdot i) && \text{by (RI0), (RI1), (RI2), (CC2)} \end{aligned}$$

Thus

$$\begin{aligned} re(t^{-1}) &= \tfrac{1}{2}(t^{-1} + \overline{t^{-1}}) \\ &= \tfrac{1}{2}((re(t) \cdot re(t) + im(t) \cdot im(t))^{-1} \cdot (re(t) - im(t) \cdot i) \\ &\quad + (re(t) \cdot re(t) + im(t) \cdot im(t))^{-1} \cdot (re(t) + im(t) \cdot i)) && \text{by (RI1), (RI2), (CC2), (CC3), (CC4)} \\ &= re(t) \cdot (re(t) \cdot re(t) + im(t) \cdot im(t))^{-1} \end{aligned}$$

Likewise

$$\begin{aligned} im(t^{-1}) &= -\tfrac{i}{2}(t^{-1} - \overline{t^{-1}}) \\ &= -\tfrac{i}{2}((re(t) \cdot re(t) + im(t) \cdot im(t))^{-1} \cdot (re(t) - im(t) \cdot i) \\ &\quad -(re(t) \cdot re(t) + im(t) \cdot im(t))^{-1} \cdot (re(t) + im(t) \cdot i)) \qquad \text{by (RI1), (RI2), (CC2), (CC3), (CC4)} \\ &= -im(t) \cdot (re(t) \cdot re(t) + im(t) \cdot im(t))^{-1} \end{aligned}$$

□

The real and the imaginary part of a complex number can be represented by so-called *real forms*.

Definition 4.2. *The set of* real forms *is defined inductively as follows.*

1. *0 and 1 are real forms,*
2. *$re(x)$ and $im(x)$ are real forms for every variable x,*
3. *if s and t are real forms then so are $-s$, s^{-1}, $s + t$ and $s \cdot t$.*

Lemma 4.3. *For each $t \in (\Sigma_m, i, ^-)$ there exist real forms t_1, t_2 such that*

$$Md + CC \vdash re(t) = t_1 \text{ and } Md + CC \vdash im(t) = t_2.$$

Proof. This follows by an easy induction on t. □

Since real and imaginary parts of complex numbers are independent, we can interpret ordinary meadow terms as complex terms while retaining provable equality.

Proposition 4.4. *Let s, t be Σ_m-terms with free variables among $\{x_0, \ldots, x_n, y_0, \ldots y_n\}$. For fresh variables $\{z_0, \ldots, z_n\}$,*

$$t^* \equiv t[x_0, \ldots, x_n := re(z_0), \ldots, re(z_n)][y_0, \ldots, y_n := im(z_0), \ldots, im(z_n)]$$

and

$$s^* \equiv s[x_0, \ldots, x_n := re(z_0), \ldots, re(z_n)][y_0, \ldots, y_n := im(z_0), \ldots, im(z_n)]$$

are real forms such that

$$Md + AEFR \vdash s = t \Rightarrow Md + CC + SSAV \vdash s^* = t^*$$

Here, we adopt the notation $[v_1, \ldots, v_n := r_1, \ldots, r_n]$ for the substitution σ with $\sigma(v_i) = r_i$ for $0 \leq i \leq n$ and $\sigma(v) = v$ for all $v \not\equiv v_i$, $0 \leq i \leq n$.

Proof. Suppose $Md + CC + SSAV \nvdash s^* = t^*$. Then there exists a $(\Sigma_m, i, ^-)$-structure $\mathcal{M} = (U, 0, 1, i, -, +, \cdot, ^{-1}, ^-)$ with $\mathcal{M} \not\models s^* = t^*$. Let $\mathcal{M}' = (U', 0_{\mathcal{M}'}, 1_{\mathcal{M}'}, -_{\mathcal{M}'}, +_{\mathcal{M}'}, \cdot_{\mathcal{M}'}, ^{-1_{\mathcal{M}'}})$ be

the Σ_m-structure obtained from $\mathcal{M}$ by stipulating $U' = \{re(u) \mid u \in U\}$, $0_{\mathcal{M}'} = 0$, $1_{\mathcal{M}'} = 1$, and for $u, u_0, u_1 \in U'$,

$$\begin{array}{rcl} -_{\mathcal{M}'}(u) & = & -u \\ +_{\mathcal{M}'}(u_0, u_1) & = & u_0 + u_1 \\ \cdot_{\mathcal{M}'}(u_0, u_1) & = & u_0 \cdot u_1 \\ (u)^{-1_{\mathcal{M}'}} & = & u^{-1}. \end{array}$$

Then $\mathcal{M}'$ is well-defined: $0_{\mathcal{M}'}, 1_{\mathcal{M}'} \in U'$ by (RI7), (RI8), and e.g. since

$$\begin{aligned} re(re(u)^{-1}) &= re(re(u)) \cdot (re(re(u)) \cdot re(re(u)) + im(re(u)) \cdot im(re(u)))^{-1} && \text{by (RI13)} \\ &= re(u) \cdot (re(u) \cdot re(u))^{-1} && \text{by (RI3), (RI5)} \\ &= re(u)^{-1} \end{aligned}$$

it follows that $(u)^{-1_{\mathcal{M}'}} \in U'$ for every $u \in U'$. Moreover, since $\mathcal{M} \not\models s^* = t^*$, $\mathcal{M}' \not\models s = t$. Clearly, $\mathcal{M}' \models Md$ and as

$$1_{1+re(x_0)\cdot re(x_0)+\cdots+re(x_n)\cdot re(x_n)} = 1_{1+re(x_0)\cdot\overline{re(x_0)}+\cdots+re(x_n)\cdot\overline{re(x_n)}} = 1$$

also $\mathcal{M}' \models AEFR$. Thus $Md + AEFR \not\vdash s = t$. □

We can now apply our previous completeness result in order to prove the axiomatization of the complex numbers to be complete.

Theorem 4.5. *Let s, t be $(\Sigma_m, i, ^-)$-terms. Then $Md + CC + SSAV \vdash s = t$ if and only if $(\mathbb{C}_0, i, ^-) \models s = t$.*

Proof. Soundness is again immediate.

Assume $(\mathbb{C}_0, i, ^-) \models s = t$. Then $(\mathbb{C}_0, i, ^-) \models s - t = 0$. By Lemma 4.3 we can pick real forms r_1, r_2 such that

$$Md + CC + SSAV \vdash re(s - t) = r_1 \text{ and } Md + CC + SSAV \vdash im(s - t) = r_2.$$

Then

$$(\mathbb{C}_0, i, ^-) \models re(s - t) = r_1 \text{ and } (\mathbb{C}_0, i, ^-) \models im(s - t) = r_2.$$

From (RI7) and (RI15) it follows that

$$(\mathbb{C}_0, i, ^-) \models r_1 = 0 \text{ and } (\mathbb{C}_0, i, ^-) \models r_2 = 0.$$

Without loss of generality, we may assume that the free variables of r_1 and r_2 are amongst $\{z_0, \ldots, z_n\}$. Now choose fresh variables $\{x_0, \ldots, x_n, y_0, \ldots y_n\}$ and consider the meadow terms u_1, u_2 obtained from r_1, r_2 by replacing occurrences of a subterm of the form $re(z_i)$ by the variable x_i and occurrences of subterms of the form $im(z_i)$ by the variable y_i. Then

$$\mathbb{R}_0 \models u_1 = 0 \text{ and } \mathbb{R}_0 \models u_2 = 0$$

and hence

$$Md + EFR \vdash u_1 = 0 \text{ and } Md + EFR \vdash u_2 = 0$$

$\mathbf{s}(1_x) = 1_x$	(S1)
$\mathbf{s}(0_x) = 0_x$	(S2)
$\mathbf{s}(-1) = -1$	(S3)
$\mathbf{s}(re(x)^{-1}) = \mathbf{s}(re(x))$	(S*4)
$\mathbf{s}(re(x) \cdot re(y)) = \mathbf{s}(re(x)) \cdot \mathbf{s}(re(y))$	(S*5)
$0_{\mathbf{s}(re(x))-\mathbf{s}(re(y))} \cdot (\mathbf{s}(re(x) + re(y)) - \mathbf{s}(re(x))) = 0$	(S*6)
$\mathbf{s}(x) = \mathbf{s}(re(x))$	(S*7)
$\overline{\mathbf{s}(x)} = \mathbf{s}(x)$	(S*8)

Table 13: The set *Signs** of axioms for the sign function for complex numbers

by Theorem 3.6. Whence

$$Md + AEFR \vdash u_1 = 0 \text{ and } Md + AEFR \vdash u_2 = 0$$

by Theorem 3.5. Since $r_1 \equiv u_1^*$ and $r_2 \equiv u_2^*$, it follows from Proposition 4.4 that

$$Md + CC + SSAV \vdash r_1 = 0 \text{ and } Md + CC + SSAV \vdash r_2 = 0.$$

Therefore

$$Md + CC + SSAV \vdash s - t = re(s - t) + im(s - t) \cdot i = 0,$$

and thus $Md + CC + SSAV \vdash s = t$. □

Remark 4.6. *One obtains a finite axiomatization of the complex numbers by replacing SSAV by the eight axioms* (S1) – (S3), (S*4) – (S*8) *for the sign function given in Table 13. Note that* $\mathbf{s}$ *differs from the usual generalizations of the sign function on* $\mathbb{R}$ *to* $\mathbb{C} - \{0\}$ *given by* $sgn(z) = z/|z|$*—where* $|z|$ *is the absolute value of* z*—and*

$$csgn(z) = \begin{cases} 1 & \text{if } re(z) > 0, \\ -1 & \text{if } re(z) < 0, \\ sgn(im(z)) & \text{if } re(z) = 0. \end{cases}$$

That $Md + CC + Signs^*$ *is indeed a complete axiomatization of the meadow of complex numbers can be seen by redoing the proof given above. Observe that* $(C0_1)$*—which is used in Proposition 4.1—follows from Signs** *by*

$$1_{\underline{2}} = \mathbf{s}(1_{\underline{2}}) = \mathbf{s}(re(1)) = \mathbf{s}(1) = 1.$$

.

References

[1] Artin, E. and Schreier, O. (1927). Algebraische Konstruktion reeller Körper. In: *Artin's Collected Papers*, eds. S. Lang and J. Tate, pages 258–272, Springer.

[2] Bergstra, J.A., Bethke, I., and Ponse, A. (2013). Cancellation meadows: a generic basis theorem and some applications. *The Computer Journal*, 56(1):3–14, `doi:10.1093/comjnl/bsx147`. (Also available at arXiv:0803.3969 [math.RA, cs.LO].)

[3] Bergstra, J.A., Hirshfeld, Y., and Tucker, J.V. (2009). Meadows and the equational specification of division. *Theoretical Computer Science*, 410(12-13):1261–1271. (Also available at arXiv:0901.0823v1 [math.RA].)

[4] Bergstra, J.A., Ponse, A., and Van der Zwaag, M.B. (2007). Tuplix calculus. *Scientific Annals of Computer Science*, 18:35-61. (Also available at arXiv:0712.3423v1 [cs.LO].)

[5] Bergstra, J.A. and Tucker, J.V. (2006). Elementary algebraic specifications of the rational complex numbers. In: *Algebra, meaning and computation. Essays dedicated to Joseph A. Goguen on the occasion of his 65th birthday*, eds. K. Futatsugi, J.-P. Jouannaud, J. Meseguer, pages 459–475, Springer Lecture Notes in Computer Science 4060, Springer.

[6] Bergstra, J.A. and Tucker, J.V. (2007). The rational numbers as an abstract data type. *Journal of the ACM*, 54(2), Article No. 7.

[7] Bethke I., Rodenburg, P.H., and Sevenster, A. (2009). The structure of finite meadows. Available at arXiv:0903.1196v1 [cs.LO].

[8] Chang, C.C. and Keisler, H.J. (1990). *Model Theory (3rd edition)*. North-Holland.

[9] Jacobson, N. (1980). *Basic Algebra II*. W.H. Freeman and Company.

[10] Poizat, B. (2000). *A Course in Model Theory. An Introduction to Contemporary Mathematical Logic*. Springer.

[11] Sander, T. (1991). Existence and uniqueness of the real closure of an ordered field without Zorn's Lemma. *Journal of Pure and Applied Algebra*, 73:165–180.

[12] Tarski, A. (1951). *A Decision Method for Elementary Algebra and Geometry, 2nd ed.* University of California Press, Berkeley, CA.

Appendix

In this appendix we will give a syntactic proof of Proposition 3.8 without the use of *IL*. We obtained this proof from a proof relying on *IL* by applying the *IL*-elimination steps suggested in the proof of Theorem 2.3. We first list some properties of pseudo ones and pseudo zeros in Table 14.

$0_t \cdot 0_t = 0_t$	(PC1)
$0_{t^2} = 0_t$	(PC2)
$0_t \cdot t = 0$	(PC3)
$0_s \cdot 0_{s+t} = 0_s \cdot 0_t$	(PC4)
$1_t \cdot 1_t = 1_t$	(PC5)
$1_{t^2} = 1_t$	(PC6)
$1_t \cdot t = t$	(PC7)
$1_s \cdot 1_t = 1_{s \cdot t}$	(PC8)

Table 14: Some properties of pseudo constants for Σ_{ms}-terms s, t

We derive (PC4): Since

$$\begin{aligned} 0_s \cdot (s+t) &= 0_s \cdot s + 0_s \cdot t \\ &= 0_s \cdot t && \text{by (PC3)} \end{aligned}$$

we have

$$\begin{aligned} 0_s^{-1} \cdot (s+t)^{-1} &= (0_s \cdot (s+t))^{-1} \\ &= (0_s \cdot t)^{-1} \\ &= 0_s^{-1} \cdot t^{-1} \end{aligned}$$

and hence

$$\begin{aligned} 0_s \cdot (s+t)^{-1} &= 0_s^2 \cdot 0_s^{-1} \cdot (s+t)^{-1} \\ &= 0_s^2 \cdot 0_s^{-1} \cdot t^{-1} \\ &= 0_s \cdot t^{-1}. \end{aligned}$$

Thus

$$\begin{aligned} 0_s \cdot 0_{s+t} &= 0_s - 0_s \cdot (s+t) \cdot (s+t)^{-1} \\ &= 0_s - 0_s \cdot t^{-1} \cdot (s+t) \\ &= 0_s - 0_s \cdot t \cdot t^{-1} \\ &= 0_s \cdot 0_t. \end{aligned}$$

For the remaining identities see [2]. Moreover, we have the following useful identity.

Lemma 4.7. *For all $n \in \mathbb{N}$, $Md + Signs \vdash 0_{x_0^2 + \cdots + x_n^2} \cdot 1_{x_0 \cdot \ \cdots \ \cdot x_n} = 0$.*

Proof. We first prove by induction on n that

$$Md + Signs \vdash 1_{x_0 \cdot \ \cdots \ \cdot x_n} \cdot \mathbf{s}(x_0^2 + \cdots + x_n^2) = 1_{x_0 \cdot \ \cdots \ \cdot x_n} \qquad (\dagger)$$

For $n = 0$ observe that $1_x \cdot \mathbf{s}(x^2) = 1_x \cdot 1_x = 1_x$ by (S7) and (PC5). Suppose $n = m + 1$. By the induction hypothesis we have

$$1_{x_0 \cdot \cdots \cdot x_m} \cdot \mathbf{s}(x_0^2 + \cdots + x_m^2) = 1_{x_0 \cdot \cdots \cdot x_m}$$

and thus $1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot \mathbf{s}(x_0^2 + \cdots + x_m^2) = 1_{x_0 \cdot \cdots \cdot x_{m+1}}$ by (PC8). It follows that

$$\begin{aligned}
0 &= 1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot \mathbf{s}(x_0^2 + \cdots + x_m^2) - 1_{x_0 \cdot \cdots \cdot x_{m+1}} \\
&= 1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot \mathbf{s}(x_0^2 + \cdots + x_m^2) - 1_{x_0 \cdot \cdots \cdot x_m} \cdot 1_{x_{m+1}} && \text{by (PC8)} \\
&= 1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot \mathbf{s}(x_0^2 + \cdots + x_m^2) - 1_{x_0 \cdot \cdots \cdot x_m} \cdot 1_{x_{m+1}} \cdot 1_{x_{m+1}} && \text{by (PC5)} \\
&= 1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot \mathbf{s}(x_0^2 + \cdots + x_m^2) - 1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot 1_{x_{m+1}} && \text{by (PC8)} \\
&= 1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot \mathbf{s}(x_0^2 + \cdots + x_m^2) - 1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot \mathbf{s}(x_{m+1}^2) && \text{by (S7)} \\
&= 1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot (\mathbf{s}(x_0^2 + \cdots + x_m^2) - \mathbf{s}(x_{m+1}^2)).
\end{aligned}$$

Therefore

$$1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot 0_{\mathbf{s}(x_0^2 + \cdots + x_m^2) - \mathbf{s}(x_{m+1}^2)} = 1_{x_0 \cdot \cdots \cdot x_{m+1}}$$

and hence by (S6), (S7) and (PC5)

$$\begin{aligned}
0 &= 1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot 0 \\
&= 1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot 0_{\mathbf{s}(x_0^2 + \cdots + x_m^2) - \mathbf{s}(x_{m+1}^2)} \cdot (\mathbf{s}(x_0^2 + \cdots + x_{m+1}^2) - \mathbf{s}(x_{m+1}^2)) \\
&= 1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot 1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot (\mathbf{s}(x_0^2 + \cdots + x_{m+1}^2) - 1_{x_0 \cdot \cdots x_{m+1}}) \\
&= 1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot \mathbf{s}(x_0^2 + \cdots + x_{m+1}^2) - 1_{x_0 \cdot \cdots \cdot x_{m+1}},
\end{aligned}$$

i.e. $1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot \mathbf{s}(x_0^2 + \cdots + x_{m+1}^2) = 1_{x_0 \cdot \cdots \cdot x_{m+1}}$.

We then have

$$\begin{aligned}
0 &= 0 \cdot 1_{x_0 \cdot \cdots \cdot x_n} \\
&= \mathbf{s}(0) \cdot 1_{x_0 \cdot \cdots \cdot x_n} \\
&= \mathbf{s}(0_{x_0^2 + \cdots + x_n^2} \cdot (x_0^2 + \cdots + x_n^2)) \cdot 1_{x_0 \cdot \cdots \cdot x_n} && \text{by (PC3)} \\
&= 0_{x_0^2 + \cdots + x_n^2} \cdot \mathbf{s}(x_0^2 + \cdots + x_n^2) \cdot 1_{x_0 \cdot \cdots \cdot x_n} && \text{by (S2), (S5)} \\
&= 0_{x_0^2 + \cdots + x_n^2} \cdot 1_{x_0 \cdot \cdots \cdot x_n} && \text{by (†).}
\end{aligned}$$

□

We now prove:

Proposition 3.8 (revisited). *For every $n \geq 0$, $Md + Signs \vdash EFR_n$.*

Proof. We prove EFR_n by induction on n. For $n = 0$ observe that $0_{x^2} \cdot x = 0_x \cdot x = 0$ by (PC2) and (PC3). For $n = m + 1$ assume that $Md + Signs \vdash 0_{y_0^2 + \cdots + y_m^2} \cdot y_0 = 0$ (IH). Then for all $0 \leq i \leq m + 1$

$$0_{x_0^2 + \cdots + x_{m+1}^2} \cdot 0_{x_i} \cdot x_0 = 0 : \qquad (‡)$$

for $i = 0$, this follows from $0_{x_0} \cdot x_0 = 0$ (PC3); for $1 \leq i \leq m+1$ we have

$$\begin{aligned}
0_{x_0^2+\cdots+x_{m+1}^2} \cdot 0_{x_i} \cdot x_0 &= 0_{x_0^2+\cdots+x_{m+1}^2} \cdot 0_{x_i^2} \cdot x_0 && \text{by (PC2)} \\
&= 0_{x_i^2} \cdot 0_{x_0^2+\cdots+x_{m+1}^2} \cdot x_0 \\
&= 0_{x_i^2} \cdot 0_{x_0^2+\cdots+x_{i-1}^2+x_{i+1}^2+\cdots+x_{m+1}^2+x_i^2} \cdot x_0 \\
&= 0_{x_i^2} \cdot 0_{x_0^2+\cdots+x_{i-1}^2+x_{i+1}^2+\cdots+x_{m+1}^2} \cdot x_0 && \text{by (PC4)} \\
&= 0 && \text{by (IH).}
\end{aligned}$$

Thus

$$\begin{aligned}
&0_{x_0^2+\cdots+x_{m+1}^2} \cdot x_0 \\
&= 0_{x_0^2+\cdots+x_{m+1}^2} \cdot (1_{x_0} + 0_{x_0}) \cdot x_0 \\
&= 0_{x_0^2+\cdots+x_{m+1}^2} \cdot 1_{x_0} \cdot x_0 && \text{by } (\ddagger) \\
&= 0_{x_0^2+\cdots+x_{m+1}^2} \cdot 1_{x_0} \cdot (1_{x_1} + 0_{x_1}) \cdot x_0 \\
&= 0_{x_0^2+\cdots+x_{m+1}^2} \cdot 1_{x_0 \cdot x_1} \cdot x_0 && \text{by } (\ddagger) \text{ and (PC8)} \\
&\vdots \\
&= 0_{x_0^2+\cdots+x_{m+1}^2} \cdot 1_{x_0 \cdot \cdots \cdot x_m} \cdot (1_{x_{m+1}} + 0_{x_{m+1}}) \cdot x_0 \\
&= 0_{x_0^2+\cdots+x_{m+1}^2} \cdot 1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot x_0 && \text{by } (\ddagger) \text{ and (PC8).}
\end{aligned}$$

We may therefore conclude that

$$\begin{aligned}
&0_{x_0^2+\cdots+x_{m+1}^2} \cdot x_0 \\
&= 0_{x_0^2+\cdots+x_{m+1}^2} \cdot (1_{x_0 \cdot \cdots \cdot x_{m+1}} + 0_{x_0 \cdot \cdots \cdot x_{m+1}}) \cdot x_0 \\
&= 0_{x_0^2+\cdots+x_{m+1}^2} \cdot 1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot x_0 + 0_{x_0^2+\cdots+x_{m+1}^2} \cdot 0_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot x_0 \\
&= 0_{x_0^2+\cdots+x_{m+1}^2} \cdot 0_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot x_0 && \text{by Lemma 4.7} \\
&= 0_{x_0^2+\cdots+x_{m+1}^2} \cdot 1_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot 0_{x_0 \cdot \cdots \cdot x_{m+1}} \cdot x_0 && \text{by the above identity} \\
&= 0 && \text{since } 1_t \cdot 0_t = 0.
\end{aligned}$$

□

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