A COMPLETE SYSTEM OF FOUR-VALUED LOGIC

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Abstract: We study a four-valued logic discussed in [BBR]. We describe a complete set of connectives containing a disjunction and a conjunction, and show the elements of the set are independent. We construct a complete set of equational axioms for the algebra of four truth values with these connectives for operations.

1. Introduction

In [BBR], a four-valued logic called $\Sigma_4(\neg,\wedge,\downarrow,\curlywedge)$ was introduced and shown to be functionally complete. As argued there, functional completeness implies, by a theorem of Bergstra and Heering [BH], the existence of a finite basis for the equational theory of the corresponding algebra of truth values. Alternatively the proof of functional completeness suggests a direct construction of a basis, and this construction forms the substance of the sequel.

As a preliminary I will show that the connectives $\curlyvee$, $\wedge$ and $\downarrow$, with the constants *divergent* and *meaningless*, are mutually independent and functionally complete.

I will deviate from the notation of [BBR] by using $\top$ and $\bot$ for the top (*meaningless*) and bottom (*divergent*) of the informativity lattice of [B], instead of *m* and *d*, and 0 and 1 for the boolean truth values *false* and *true* instead of *f* and *t*.

2. An independent system of connectives

Let *L* be the set $\{\bot, \top, 0, 1\}$ of four distinct truth values. The binary operations $\curlyvee$ (left-first disjunction) and $\wedge$ (parallel conjunction), and the unary operation $\downarrow$ (convergence test) are defined by the following tables:

$\curlyvee$	$\bot$	$\top$	0	1
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$
0	$\bot$	$\top$	0	1
1	1	1	1	1

$\wedge$	$\bot$	$\top$	0	1
$\bot$	$\bot$	$\top$	0	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$
0	0	$\top$	0	0
1	$\bot$	$\top$	0	1

x	$\downarrow$x
$\bot$	0
$\top$	0
0	1
1	1

Let **L** be the algebra $\langle L, \veebar, \wedge, \downarrow, \bot, \top\rangle$.

We construct formal *terms* from variables **x**, **y**,…, $\mathbf{x}_1$, $\mathbf{x}_2$,…, and constant symbols $\bot$ and $\top$ by means of operation symbols $\veebar$, $\wedge$, $\downarrow$, as usual. Greek letters φ, ψ,… will be used as metavariables over terms, and the interpretation of φ in **L** will be denoted by φ.

2.1 Lemma. Suppose φ is a term built from one variable **x** by means of the connectives $\wedge$, $\downarrow$, $\bot$, and $\top$. If $\varphi(a) = \top$ for some $a \in L - \{\top\}$, then φ is constant.

Proof. By induction on the construction of φ. The basic terms **x**, $\bot$, and $\top$ do not satisfy the condition; neither does any term beginning with $\downarrow$. If $\varphi(a) \wedge \psi(a) = \top$, then $\varphi(a) = \top$ or $\psi(a) = \top$. Say $\varphi(a) = \top$, without losing generality. Then φ is constantly $\top$, by induction hypothesis; so $\varphi \wedge \psi$ is constant. ¶

Corollary. The connective $\veebar$ is not definable from $\{\wedge, \downarrow, \bot, \top\}$.

Proof. Consider $\mathbf{x} \veebar \top$; it does not have the property of the lemma (it is not constant, and it has value $\top$ on the argument 0). ¶

Let $\mathbf{L}_0$ be the reduct of **L** obtained by removal of $\wedge$.

2.2 Lemma. Identification of $\bot$ and $\top$ is a congruence of $\mathbf{L}_0$.

Proof. By checking all cases of the substitution condition. For example, if $a \veebar \bot \in \{\bot,\top\}$, then either $a \in \{\bot,\top\}$ or $a = 0$, and then $a \veebar \top \in \{\bot,\top\}$ as well. ¶

Corollary. The connective $\wedge$ is not definable from $\{\veebar, \downarrow, \bot, \top\}$.

Proof. If $\wedge$ were definable, identification of $\bot$ and $\top$ should be a congruence of **L**; but it is not, since $0 \wedge \bot = 0$ and $0 \wedge \top = \top$. ¶

2.3 Lemma. Suppose φ is a term built from one variable **x** by means of the connectives $\veebar$, $\wedge$, $\bot$, and $\top$. If $\varphi(\top) \neq \top$, then φ is constant.

Proof. By induction on the construction of φ. The basic terms **x** and $\top$ do not satisfy the condition; $\bot$ is constant. If $\varphi(\top) \veebar \psi(\top) \neq \top$, then $\varphi(\top) \neq \top$, so φ is constant by induction hypothesis. If φ is 1, $\bot$, or $\top$, then $\varphi \veebar \psi = \varphi$. If $\varphi = 0$, then $\varphi \veebar \psi = \psi$, and ψ is constant by induction hypothesis.

Finally, if $\varphi(\top) \wedge \psi(\top) \neq \top$, then $\varphi(\top) \neq \top$ and $\psi(\top) \neq \top$, so φ and ψ are constant by induction hypothesis; so $\varphi \wedge \psi$ is constant. ¶

Corollary. The connective $\downarrow$ is not definable from $\{\mathring{\vee}, \wedge, \bot, \top\}$.

Proof. $\downarrow\top \neq \top$, but $\downarrow\mathbf{x}$ is not constant. ¶

2.4 Lemma. Let φ be a term built from a variable **x** by means of the connectives $\mathring{\vee}$, $\wedge$, and $\downarrow$. Then $\varphi(\bot) \neq \top$ and $\varphi(\top) \neq \bot$.

Proof. Trivial induction on the construction of φ. ¶

To summarize the lemmas and their corollaries:

2.5 Theorem (*independence of connectives*) None of the operations of **L** is definable in terms of the others.

3. Some definable connectives

The most conspicuous missing connective is negation. We put:

$$\neg\mathbf{x} := \downarrow(\mathbf{x} \wedge \bot) \mathring{\vee} (\downarrow(\downarrow\mathbf{x} \wedge \bot) \wedge \mathbf{x}).$$

By straightforward calculation we obtain the truth table (the same as in [BBR]):

x	¬ x
⊥	⊥
⊤	⊤
0	1
1	0

Next we define left-first conjunction:

$$\mathbf{x} \mathring{\wedge} \mathbf{y} := \neg(\neg\mathbf{x} \mathring{\vee} \neg\mathbf{y}).$$

Again the truth table is as in [BBR]:

$\mathring{\wedge}$	⊥	⊤	0	1
⊥	⊥	⊥	⊥	⊥
⊤	⊤	⊤	⊤	⊤
0	0	0	0	0
1	⊥	⊤	0	1

Thus, $\curlywedge$, $\wedge$, $\neg$, $\downarrow$, $\bot$, and $\top$, are definable operations of **L**. So by Theorem 3.1 of [BBR], **L** is functionally complete:

3.1 Proposition. Every finitary operation on the set L is definable in the algebra **L**.

The boolean constants are defined by

$$0 := \downarrow\top; \qquad 1 := \downarrow 0.$$

Parallel disjunction may be defined by

$$\mathbf{x} \vee \mathbf{y} := \neg(\neg\mathbf{x} \wedge \neg\mathbf{y}).$$

We shall use tests for falsity, truth, divergence, and meaninglessness:

$$\mathbf{Fx} := \downarrow(\mathbf{x} \wedge \bot);\ \mathbf{Tx} := \mathbf{F}\neg\mathbf{x};\ \mathbf{Dx} := \mathbf{F}\downarrow\mathbf{x} \wedge \downarrow(0 \wedge \mathbf{x});\ \mathbf{Mx} := \mathbf{F}\downarrow(0 \wedge \mathbf{x}).$$

The truth tables for these operations are

$\vee$	$\bot$	$\top$	0	1
$\bot$	$\bot$	$\top$	$\bot$	1
$\top$	$\top$	$\top$	$\top$	$\top$
0	$\bot$	$\top$	0	1
1	1	$\top$	1	1

x	Fx
$\bot$	0
$\top$	0
0	1
1	0

x	Tx
$\bot$	0
$\top$	0
0	0
1	1

x	Dx
$\bot$	1
$\top$	0
0	0
1	0

x	Mx
$\bot$	0
$\top$	1
0	0
1	0

All these operations also occur in [BBR], but not always with the same definitions. Instead of **D**, **M**, **F**, and **T**, [BBR] used the letter κ with subscripts. The same notation will be used here: I write κ_a, with a varying over L. So $\kappa_\bot$ is **D**, etc.

Observe that $\neg\mathbf{x} = \mathbf{Fx} \curlyvee (\mathbf{F}\downarrow\mathbf{x} \wedge \mathbf{x})$.

The *nonconvergence test* we define by $\uparrow\mathbf{x} = \mathbf{F}\downarrow\mathbf{x}$.

The algebra **L** satisfies the following duality principle: for a term φ, let φ^d be the result of simultaneously replacing every occurrence of $\downarrow$ by $\uparrow$, $\wedge$ by $\vee$, and $\curlyvee$ by $\curlywedge$; then for all terms φ, ψ,

$$\mathbf{L} \vDash \varphi \equiv \psi \text{ if and only if } \mathbf{L} \vDash \varphi^d \equiv \psi^d.$$

4. Axioms and theorems

We are looking for a complete axiomatization of the equational theory of **L**. Axioms will be introduced in clusters.

4.1 The $\curlyvee$-system

Consider the following system ($\curlyvee$) of axioms for the left-first disjunction $\curlyvee$ and the constant 0:

(1) $\mathbf{x} \curlyvee (\mathbf{y} \curlyvee \mathbf{z}) \equiv (\mathbf{x} \curlyvee \mathbf{y}) \curlyvee \mathbf{z}$ (associative law)

(2) $\mathbf{x} \curlyvee \mathbf{y} \equiv \mathbf{x} \curlyvee (\mathbf{y} \curlyvee \mathbf{x})$ (law of repetition)

(3) $0 \curlyvee \mathbf{x} \equiv \mathbf{x}$ (unit law)

4.1.1 Proposition. The equations (1-3) are valid in **L**.

Convention. Let A be a set of equations; then I write $A \vdash \varphi \equiv \psi$ to communicate that the equation $\varphi \equiv \psi$ is deducible from A by equational logic. The part 'A $\vdash$' will often be omitted when it is clear from the context.

4.1.2 Lemma. $(\curlyvee) \vdash \mathbf{x} \curlyvee 0 \equiv \mathbf{x}$.

Proof. $\mathbf{x} \curlyvee 0 \equiv 0 \curlyvee (\mathbf{x} \curlyvee 0) \equiv 0 \curlyvee \mathbf{x} \equiv \mathbf{x}$ by (3) and (2). ¶

For terms φ, we define Focφ to be the sequence of variables in φ in order of first occurrence. E.g.,

$$\mathrm{Foc}((\mathbf{x} \wedge \mathbf{y}) \curlyvee (\mathbf{x} \wedge \mathbf{z})) = \langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle.$$

4.1.3 Theorem. Let φ and ψ be built from variables by the sole means of the connective $\curlyvee$. Then $(\curlyvee) \vdash \varphi \equiv \psi$ iff Focφ = Focψ.

Proof.
($\Rightarrow$) Suppose $\mathrm{Foc}\varphi = \langle \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle$, $\mathrm{Foc}\psi = \langle \mathbf{y}_1, \ldots, \mathbf{y}_m \rangle$, and $\mathrm{Foc}\varphi \neq \mathrm{Foc}\psi$. If there exists $j \leq m$, n such that $\mathbf{x}_j \neq \mathbf{y}_j$, take the least such j (say it is k), and let $x_k = \top$ and $y_k = 1$; assign 0 to the other variables. Then $\varphi(x_1, \ldots, x_n) = \top$ and $\psi(y_1, \ldots, y_m) = 1$; since $\mathbf{L} \models (\curlyvee)$, this proves that $(\curlyvee) \nvdash \varphi \equiv \psi$. If on the other hand there is no $j \leq m$, n such that $\mathbf{x}_j \neq \mathbf{y}_j$, there must be a variable $\mathbf{z}$ in one of φ and ψ that does not occur in the other. Then assign $\top$ (or 1, or $\bot$) to $\mathbf{z}$, and 0 to all the other variables, and argue analogously.
($\Leftarrow$) For a sequence $\langle \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle$ of variables ($n \geq 1$), let $\curlyvee \mathbf{x}_1 \ldots \mathbf{x}_n$ be the repeated left-first disjunction $\mathbf{x}_1 \curlyvee (\mathbf{x}_2 \curlyvee (\mathbf{x}_3 \curlyvee (\ldots (\mathbf{x}_{n-1} \curlyvee \mathbf{x}_n) \ldots)))$ (associating to the right). We set $\curlyvee \varnothing = 0$. It will suffice to prove that for all φ constructed from $\curlyvee$ and variables, $(\curlyvee) \vdash \varphi \equiv \curlyvee \mathrm{Foc}\varphi$. We proceed by induction on the number N of variable occurrences in φ.

If $N = 1$, there is nothing to prove.

Now suppose $N > 1$; we assume that $(\curlyvee) \vdash \psi \equiv \curlyvee\mathrm{Foc}\psi$ for all left-first disjunctions ψ containing fewer than N variable occurrences. Now φ has the form $\varphi_1 \curlyvee \varphi_2$, and by induction hypothesis

$$(\curlyvee) \vdash \varphi \equiv \curlyvee\mathrm{Foc}\varphi_1 \curlyvee \curlyvee\mathrm{Foc}\varphi_2.$$

If φ_1 and φ_2 have no variables in common, we get $\varphi \equiv \curlyvee\mathrm{Foc}\varphi$ by associating to the right, using (1). Otherwise, let $\mathrm{Foc}\varphi_2 = \langle \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle$ $(n \geq 1)$, and suppose i is the first index j such that $\mathbf{x}_j$ occurs in φ_1. By induction hypothesis,

$$\curlyvee\mathrm{Foc}\varphi_2 \equiv (\curlyvee\mathbf{x}_1 \ldots \mathbf{x}_{i-1} \curlyvee \mathbf{x}_i) \curlyvee \curlyvee\mathbf{x}_{i+1} \ldots \mathbf{x}_n.$$

Suppose $\mathrm{Foc}\varphi_1 = \langle \bar{\mathbf{y}}, \mathbf{x}_i, \bar{\mathbf{z}} \rangle$. If $\bar{\mathbf{y}}$ is nonempty,

$$\begin{aligned}
\curlyvee\mathrm{Foc}\varphi_1 \curlyvee \curlyvee\mathrm{Foc}\varphi_2 &\equiv (\curlyvee\bar{\mathbf{y}} \curlyvee (\mathbf{x}_i \curlyvee \curlyvee\bar{\mathbf{z}})) \curlyvee \curlyvee\mathrm{Foc}\varphi_2 && \text{(ind. hyp.)} \\
&\equiv \curlyvee\bar{\mathbf{y}} \curlyvee ((\mathbf{x}_i \curlyvee \curlyvee\bar{\mathbf{z}}) \curlyvee \curlyvee\mathrm{Foc}\varphi_2) && (1) \\
&\equiv \curlyvee\bar{\mathbf{y}} \curlyvee \curlyvee\mathrm{Foc}((\mathbf{x}_i \curlyvee \curlyvee\bar{\mathbf{z}}) \curlyvee \curlyvee\mathrm{Foc}\varphi_2) && \text{(ind. hyp.)} \\
&\equiv \curlyvee\mathrm{Foc}\varphi && \text{by ind. hyp.,}
\end{aligned}$$

since we have suppressed an occurrence of $\mathbf{x}_i$. A similar argument applies if $\bar{\mathbf{y}}$ is empty and $i < n$. If $\bar{\mathbf{y}}$ is empty and $i = n$, we have

$$\begin{aligned}
\curlyvee\mathrm{Foc}\varphi_1 \curlyvee \curlyvee\mathrm{Foc}\varphi_2 &\equiv (\mathbf{x}_i \curlyvee \curlyvee\bar{\mathbf{z}}) \curlyvee (\curlyvee\mathbf{x}_1 \ldots \mathbf{x}_{i-1} \curlyvee \mathbf{x}_i) && \text{(ind. hyp.)} \\
&\equiv \mathbf{x}_i \curlyvee ((\curlyvee\bar{\mathbf{z}} \curlyvee \curlyvee\mathbf{x}_1 \ldots \mathbf{x}_{i-1}) \curlyvee \mathbf{x}_i) && \text{((1), twice)} \\
&\equiv \mathbf{x}_i \curlyvee (\curlyvee\bar{\mathbf{z}} \curlyvee \curlyvee\mathbf{x}_1 \ldots \mathbf{x}_{i-1}) && (2) \\
&\equiv \curlyvee\mathrm{Foc}\varphi && \text{by ind. hyp.} \quad ¶
\end{aligned}$$

We have shown in particular that the operation $\curlyvee$ is idempotent.

Of course, if \$ is a substitution and φ and ψ are as in the statement of theorem 4.1.3, we have $(\curlyvee) \vdash \$\varphi \equiv \$\psi$; this is the form in which the theorem will be used. Moreover it allows us to write repeated left-first disjunctions without parentheses.

By the unit law, equations of the form $\varphi \curlyvee \psi \equiv \varphi \curlyvee \chi$ are *conditionals*: they may be taken to state that $\$\psi \equiv \$\chi$, for any substitution \$, under the condition that $\$\varphi \equiv 0$.

4.2 The $\wedge$-system

Next we have the following system $(\wedge)$ of axioms for the parallel conjunction $\wedge$ and the constants T and 1:

(1) $\mathbf{x} \wedge (\mathbf{y} \wedge \mathbf{z}) \equiv (\mathbf{x} \wedge \mathbf{y}) \wedge \mathbf{z}$ (associative law)

(2) $\mathbf{x} \wedge \mathbf{y} \equiv \mathbf{y} \wedge \mathbf{x}$ (commutative law)

(3) $\mathbf{x} \wedge \mathbf{x} \equiv \mathbf{x}$ (idempotency law)

(4) $\mathbf{x} \wedge \top \equiv \top$ (zero law)

(5) $\mathbf{x} \wedge 1 \equiv \mathbf{x}$ (unit law)

4.2.1 Proposition. $\mathbf{L} \models (\wedge)$.

The system $(\wedge)$ is well known; it defines the infimum-semilattices with least element $\top$ and greatest element 1. For a term φ, let us denote by $\mathrm{Var}\varphi$ the set of all variables that occur in φ.

4.2.2 Proposition. Let φ and ψ be built from variables by the sole means of the connective $\wedge$. Then $(\wedge) \vdash \varphi \equiv \psi$ iff $\mathrm{Var}\varphi = \mathrm{Var}\psi$.

As with 4.1.3, we shall freely use substitution instances of this proposition. We write repeated parallel conjunctions without parentheses. We also use the infimum sign $\bigwedge$ for repeated $\wedge$; $\bigwedge\varnothing = 1$ in accordance with (5).

4.3 The $\curlyvee\wedge$-system

The system $(\curlyvee\wedge)$ consists of $(\curlyvee) \cup (\wedge)$ extended with

(*) $\mathbf{x} \curlyvee (\mathbf{y} \wedge \mathbf{z}) \equiv (\mathbf{x} \curlyvee \mathbf{y}) \wedge (\mathbf{x} \curlyvee \mathbf{z})$

Proposition. $\mathbf{L} \models (\curlyvee\wedge)$.

4.4 Basic test-axioms

On top of $(\curlyvee\wedge)$ comes the system $(\downarrow)$ of unconditional axioms involving $\downarrow$:

(1) $\downarrow\downarrow\mathbf{x} \equiv 1$ (double test)

(2) $\downarrow\bot \wedge \bot \equiv 0$ (out of queue)

(3) $\downarrow\mathbf{x} \wedge \uparrow\mathbf{x} \equiv 0$ (contradiction)

(4) $\uparrow\mathbf{x} \curlyvee \mathbf{x} \equiv \mathbf{FF}\mathbf{x}$ (double falsity)

4.4.1 Proposition. $\mathbf{L} \models (\downarrow)$.

4.4.2 Lemma. The following equations are derivable from $(\curlyvee\wedge) \cup (\downarrow)$:
(a) $\downarrow\bot \equiv \downarrow\top$ (= 0); $\downarrow 1 \equiv \downarrow 0$ (= 1);
(b) $0 \wedge \bot \equiv \downarrow\mathbf{x} \wedge 0 \equiv 0$;
(c) $\uparrow\top \equiv \uparrow\bot \equiv 1$; $\uparrow 0 \equiv \uparrow 1 \equiv 0$;

(d) $\mathbf{M}{\downarrow}\mathbf{x} \equiv \mathbf{D}{\downarrow}\mathbf{x} \equiv {\uparrow}{\downarrow}\mathbf{x} \equiv 0$;
(e) $\neg{\downarrow}\mathbf{x} \equiv {\uparrow}\mathbf{x}$; $\mathbf{F}{\uparrow}\mathbf{x} \equiv \mathbf{T}{\downarrow}\mathbf{x} \equiv {\downarrow}\mathbf{x}$;
(f) for any $a, b \in L$, $\kappa_a(\mathbf{b}) \equiv 1$ if $a = b$, and otherwise $\kappa_a(\mathbf{b}) \equiv 0$;
(g) $\neg\bot \equiv \bot$; $\neg\top \equiv \top$; $\neg 0 \equiv 1$; $\neg 1 \equiv 0$.

Proof.
(a) By ($\wedge$5) and ($\downarrow$2), ${\downarrow}\bot \equiv {\downarrow}\bot \wedge {\downarrow}0 \equiv {\downarrow}\bot \wedge {\downarrow}({\downarrow}\bot \wedge \bot)$; so by ($\downarrow$3), ${\downarrow}\bot \equiv 0$. By ($\downarrow$1), ${\downarrow}1 \equiv 1$.
(b) By (a) and ($\downarrow$2), $0 \wedge \bot \equiv {\downarrow}\bot \wedge \bot \equiv 0$. By ($\downarrow$3),

$$\downarrow\mathbf{x} \wedge 0 \equiv \downarrow\mathbf{x} \wedge \downarrow\mathbf{x} \wedge \uparrow\mathbf{x} \equiv \downarrow\mathbf{x} \wedge \uparrow\mathbf{x} \equiv 0.$$

(c) By (a) and ($\downarrow$2), ${\uparrow}\top = \mathbf{F}{\downarrow}\top \equiv \mathbf{F}{\downarrow}\bot \equiv {\downarrow}({\downarrow}\bot \wedge \bot) \equiv 1$. By (a) and ($\wedge$), ${\uparrow}1 = \mathbf{F}{\downarrow}1 \equiv \mathbf{F}{\downarrow}0 = {\downarrow}({\downarrow}0 \wedge \bot) \equiv {\downarrow}\bot \equiv 0$.
(d) $\mathbf{M}{\downarrow}\mathbf{x} = \mathbf{F}{\downarrow}(0 \wedge {\downarrow}\mathbf{x})$ by definition. It follows from (b) that ${\downarrow}(0 \wedge {\downarrow}\mathbf{x}) \equiv {\downarrow}0 = 1$. So by ($\wedge$) and (a), $\mathbf{M}{\downarrow}\mathbf{x} \equiv {\downarrow}(1 \wedge \bot) \equiv {\downarrow}\bot \equiv 0$.
By ($\downarrow$1) and (c), ${\uparrow}{\downarrow}\mathbf{x} = \mathbf{F}{\downarrow}{\downarrow}\mathbf{x} \equiv \mathbf{F}1 = {\uparrow}0 \equiv 0$.
Finally, $\mathbf{D}{\downarrow}\mathbf{x} = {\uparrow}{\downarrow}\mathbf{x} \wedge {\downarrow}(0 \wedge {\downarrow}\mathbf{x})$ by definition. By (b), ${\downarrow}(0 \wedge {\downarrow}\mathbf{x}) \equiv 1$, so $\mathbf{D}{\downarrow}\mathbf{x} \equiv 0 \wedge 1 \equiv 0$.
(e) $\neg{\downarrow}\mathbf{x} = {\uparrow}\mathbf{x} \veebar ({\uparrow}{\downarrow}\mathbf{x} \wedge {\downarrow}\mathbf{x})$ by definition. By (d) and (b), ${\uparrow}{\downarrow}\mathbf{x} \wedge {\downarrow}\mathbf{x} \equiv 0$. So $\neg{\downarrow}\mathbf{x} \equiv {\uparrow}\mathbf{x}$ by 4.1.2.
By ($\downarrow$4), (d), and ($\veebar$3), $\mathbf{F}{\uparrow}\mathbf{x} \equiv {\uparrow}{\downarrow}\mathbf{x} \veebar {\downarrow}\mathbf{x} \equiv {\downarrow}\mathbf{x}$.
$\mathbf{T}{\downarrow}\mathbf{x} = \mathbf{F}\neg{\downarrow}\mathbf{x}$ by definition. So $\mathbf{T}{\downarrow}\mathbf{x} \equiv \mathbf{F}{\uparrow}\mathbf{x} \equiv {\downarrow}\mathbf{x}$.
(f) By (b), $\mathbf{F}0 \equiv {\downarrow}0 = 1$. By 4.2 and (a), $\mathbf{F}\bot \equiv {\downarrow}\bot \equiv 0$. By ($\wedge$4), $\mathbf{F}\top = {\downarrow}(\top \wedge \bot) \equiv {\downarrow}\top = 0$. By (c), $\mathbf{F}1 \equiv {\uparrow}0 \equiv 0$.
By (d), $\mathbf{M}0 \equiv \mathbf{M}1 \equiv \mathbf{D}0 \equiv \mathbf{D}1 \equiv 0$. By (b), $\mathbf{M}\bot = \mathbf{F}{\downarrow}(0 \wedge \bot) \equiv \mathbf{F}1 \equiv 0$.
By ($\wedge$4), $\mathbf{M}\top = \mathbf{F}{\downarrow}(0 \wedge \top) \equiv \mathbf{F}0 \equiv 1$. By (c) and (b), $\mathbf{D}\bot = {\uparrow}\bot \wedge {\downarrow}(0 \wedge \bot) \equiv 1 \wedge 1 \equiv 1$. By (c) and ($\wedge$), $\mathbf{D}\top = {\uparrow}\top \wedge {\downarrow}(0 \wedge \top) \equiv 1 \wedge 0 \equiv 0$.
By (e), $\mathbf{T}1 \equiv 1$ and $\mathbf{T}0 \equiv 0$. Since by (c),

$$\neg\bot = \mathbf{F}\bot \veebar ({\uparrow}\bot \wedge \bot) \equiv 0 \veebar (1 \wedge \bot) \equiv \bot,$$

$\mathbf{T}\bot \equiv \mathbf{F}\bot \equiv 0$. Since similarly

$$\neg\top \equiv \top,$$

likewise $\mathbf{T}\top \equiv 0$.
(g) The first two equations have just been proved. By (e) and (c), $\neg 0 \equiv {\uparrow}\top \equiv 1$. By (e) and (d), $\neg 1 \equiv {\uparrow}{\downarrow}\top \equiv 0$. ¶

4.4.3 Lemma. The following equations are derivable from ($\veebar\wedge$) $\cup$ ($\downarrow$):
(a) $\mathbf{D}({\downarrow}\mathbf{x} \wedge \bot) \equiv {\downarrow}\mathbf{x}$; (b) $\mathbf{FF}\mathbf{x} \veebar \mathbf{x} \equiv \mathbf{FF}\mathbf{x}$.

Proof. (a) By 4.4.2(b, e),

$$\mathbf{D}(\downarrow\mathbf{x} \wedge \bot) = \uparrow(\downarrow\mathbf{x} \wedge \bot) \wedge \downarrow(0 \wedge \downarrow\mathbf{x} \wedge \bot) \equiv \mathbf{F}\uparrow\mathbf{x} \wedge 1 \equiv \downarrow\mathbf{x}.$$

(b) By ($\downarrow$4) and 4.2, $\mathbf{FFx} \equiv \uparrow\mathbf{x} \veebar \mathbf{x} \equiv \uparrow\mathbf{x} \vee \mathbf{x} \veebar \mathbf{x} \equiv \mathbf{FFx} \veebar \mathbf{x}$. ¶

4.5 Test distribution axioms

The following pair (D) of axioms describes the distribution of the test over the binary connectives:

(1) $\quad \downarrow(\mathbf{x} \veebar \mathbf{y}) \equiv \mathbf{Tx} \veebar (\mathbf{Fx} \wedge \downarrow\mathbf{y})$

(2) $\quad \downarrow(\mathbf{x} \wedge \mathbf{y}) \equiv (\mathbf{Fx} \wedge \mathbf{Dy}) \veebar (\mathbf{Dx} \wedge \mathbf{Fy}) \veebar (\downarrow\mathbf{x} \wedge \downarrow\mathbf{y})$

4.5.1 Proposition. $\mathbf{L} \models$ (D).

4.5.2 Lemma. The following equations are derivable from ($\veebar\wedge$) $\cup$ ($\downarrow$) $\cup$ (D):

(1) $\mathbf{Tx} \veebar \mathbf{Fx} \equiv \downarrow\mathbf{x}$ (2) $\mathbf{Tx} \vee \downarrow\mathbf{x} \equiv \downarrow\mathbf{x}$

(3) $\downarrow(\mathbf{x} \veebar \mathbf{y}) \equiv \downarrow\mathbf{x} \wedge (\mathbf{Tx} \vee \downarrow\mathbf{y})$

Proof.
(1) By ($\wedge$5) and (D1), substituting 0 for **y**,

$$\mathbf{Tx} \veebar \mathbf{Fx} \equiv \mathbf{Tx} \veebar (\mathbf{Fx} \wedge 1) \equiv \downarrow(\mathbf{x} \veebar 0) \equiv \downarrow\mathbf{x}.$$

(2) By (1) and 4.1.3, $\mathbf{Tx} \vee \downarrow\mathbf{x} \equiv \mathbf{Tx} \vee \mathbf{Tx} \veebar \mathbf{Fx} \equiv \mathbf{Tx} \vee \mathbf{Fx} \equiv \downarrow\mathbf{x}$.
(3) By (D1), ($\veebar\wedge$), and (1) and (2),

$\downarrow(\mathbf{x}\veebar\mathbf{y}) \equiv \mathbf{Tx}\veebar(\mathbf{Fx}\wedge\downarrow\mathbf{y}) \equiv (\mathbf{Tx}\vee\mathbf{Fx}) \wedge (\mathbf{Tx}\veebar\downarrow\mathbf{y}) \equiv \downarrow\mathbf{x} \wedge (\mathbf{Tx}\vee\downarrow\mathbf{y})$. ¶

4.6 Evaluation axioms

The cluster (E) of evaluation axioms consists of

(1) $\quad \mathbf{T}(\mathbf{x} \veebar \top) \equiv \mathbf{Tx}$

(2) $\quad \mathbf{D}(\mathbf{x} \veebar \top) \equiv \mathbf{Dx}$

(3) $\quad \mathbf{T}(\mathbf{x} \wedge \bot) \equiv 0$ (in queue).

4.6.1 Proposition. $\mathbf{L} \models$ (E).

4.7 Conditional axioms

The last cluster of axioms consists of conditionals: statements of the form

$$\varphi_1 \veebar \ldots \vee \varphi_n \veebar \psi \equiv \varphi_1 \veebar \ldots \vee \varphi_n \veebar \chi.$$

The items (1)-(4) are typically used to conclude that $\$\psi \equiv \$\chi$, for some substitution $\$$, from a proof that $\$\varphi_1 \equiv \ldots \equiv \$\varphi_n \equiv 0$ ('the threshold vanishes'). We refer to this cluster as (C).

(1) $\uparrow\mathbf{x} \veebar \uparrow\mathbf{y} \veebar \mathbf{x} \veebar \mathbf{y} \equiv \uparrow\mathbf{x} \veebar \uparrow\mathbf{y} \veebar \mathbf{y} \veebar \mathbf{x}$

(2) Let $\varphi = (\mathbf{Dx} \wedge \mathbf{Fy} \wedge \mathbf{Tz}) \veebar (\mathbf{Fx} \wedge \mathbf{Dy} \wedge \mathbf{Tz})$:

$$\varphi \veebar (\mathbf{x} \wedge \mathbf{y}) \veebar \mathbf{z} \equiv \varphi \veebar ((\mathbf{x} \veebar \mathbf{z}) \wedge (\mathbf{y} \veebar \mathbf{z}))$$

(3) Let $\varphi = \mathbf{Fx} \wedge (\mathbf{Dy} \veebar \mathbf{Ty}) \wedge \mathbf{Mz}$:

$$\varphi \veebar (\mathbf{x} \wedge (\mathbf{y} \veebar \mathbf{z})) \equiv \varphi \veebar (\mathbf{x} \wedge \mathbf{y}) \veebar (\mathbf{x} \wedge \mathbf{z})$$

(4) $\mathbf{My} \veebar (\mathbf{x} \veebar (\mathbf{x} \wedge \mathbf{y})) \equiv \mathbf{My} \veebar \mathbf{x}$

(5) $\downarrow\mathbf{x} \veebar \mathbf{Mx} \veebar \mathbf{x} \equiv \downarrow\mathbf{x} \veebar \mathbf{Mx} \veebar \bot$

(6) $\downarrow(0 \wedge \mathbf{x}) \veebar \mathbf{x} \equiv \downarrow(0 \wedge \mathbf{x}) \veebar \top$

(7) $\mathbf{Fx} \veebar \mathbf{x} \veebar \mathbf{y} \equiv \mathbf{Fx} \veebar \mathbf{x}$

4.7.1 Proposition. $\mathbf{L} \vDash$ (C).

4.7.2 Lemma. (a) $(\veebar\wedge) \cup (\downarrow) \cup (\mathrm{D}) \cup (\mathrm{C}) \vdash \downarrow\mathbf{x} \veebar \uparrow\mathbf{x} \equiv \downarrow\mathbf{x} \veebar 1 \equiv 1$.
(b) If $\mathbf{c} \in \{1, \bot, \top\}$, then $(\veebar\wedge) \cup (\downarrow) \cup (\mathrm{C}) \vdash \mathbf{c} \veebar \mathbf{x} \equiv \mathbf{c}$.

Proof. (a) By (C1) and 4.1,

$$\uparrow\downarrow\mathbf{x} \veebar \uparrow\uparrow\mathbf{x} \veebar \downarrow\mathbf{x} \veebar \uparrow\mathbf{x} \equiv \uparrow\downarrow\mathbf{x} \veebar \uparrow\uparrow\mathbf{x} \veebar \uparrow\mathbf{x} \veebar \downarrow\mathbf{x}.$$

But $\uparrow\downarrow\mathbf{x} \equiv \uparrow\uparrow\mathbf{x} \equiv 0$ by 4.4.2(d); so $\downarrow\mathbf{x} \veebar \uparrow\mathbf{x} \equiv \uparrow\mathbf{x} \veebar \downarrow\mathbf{x} \equiv 1$ by 4.5.2(5). Hence by 4.1, $\downarrow\mathbf{x} \veebar 1 \equiv \downarrow\mathbf{x} \veebar \downarrow\mathbf{x} \veebar \uparrow\mathbf{x} \equiv \downarrow\mathbf{x} \veebar \uparrow\mathbf{x} \equiv 1$.
(b) By 4.4.2(f), $\mathbf{Fc} \equiv 0$. So by ($\veebar$3) and (C7),

$$\mathbf{c} \veebar \mathbf{x} \equiv \mathbf{Fc} \veebar \mathbf{c} \veebar \mathbf{x} \equiv \mathbf{Fc} \veebar \mathbf{c} \equiv \mathbf{c}. \quad ¶$$

We shall refer to the entire system $(\veebar\wedge) \cup (\downarrow) \cup (\mathrm{D}) \cup (\mathrm{E}) \cup (\mathrm{C})$ of axioms as $\mathcal{L}$.

4.7.3 Definition. A *test atom* is a term with main operator $\downarrow$. A *test molecule* is a term built from test atoms by $\wedge$ and $\veebar$.

4.7.4 Lemma. If φ is a test molecule, then $\mathcal{L} \vdash \downarrow\varphi \equiv 1$, and

$$\mathcal{L} \vdash \uparrow\varphi \equiv \mathbf{D}\varphi \equiv \mathbf{M}\varphi \equiv 0 \wedge \varphi \equiv 0\,.$$

Proof. By simultaneous induction over φ. The case that φ is a test atom is immediate by ($\downarrow$1) and 4.4.2.

$\varphi = \psi\wedge\chi$: By (D2), $\downarrow\varphi \equiv (\mathbf{F}\psi \wedge \mathbf{D}\chi) \veebar (\mathbf{D}\psi \wedge \mathbf{F}\chi) \veebar (\downarrow\psi \wedge \downarrow\chi)$. Use the induction hypothesis: $\mathbf{D}\chi\equiv 0$, so $\mathbf{F}\psi\wedge\mathbf{D}\chi \equiv 0$, and similarly $\mathbf{D}\psi\wedge\mathbf{F}\chi \equiv 0$; and $\downarrow\psi\equiv\downarrow\chi\equiv 1$. By 4.4.2, $\uparrow\varphi=\mathbf{F}\downarrow\varphi\equiv\mathbf{F}1\equiv 0$. That $0\wedge\varphi \equiv 0$ is an immediate consequence of the induction hypothesis. Now

$$\mathbf{D}\varphi = \uparrow\varphi \wedge \downarrow(0\wedge\varphi) \equiv 0\wedge\downarrow 0 \equiv 0;$$

and $\mathbf{M}\varphi \equiv \uparrow(0\wedge\varphi) \equiv \uparrow 0 \equiv 0$ by 4.4.2.

$\varphi = \psi\veebar\chi$: By (D1), $\downarrow\varphi \equiv \mathbf{T}\psi \veebar (\mathbf{F}\psi \wedge \downarrow\chi)$. Use the induction hypothesis and 4.5.2(1). Now $\uparrow\varphi\equiv 0$ as before.

Since $\mathbf{D}\psi \equiv \mathbf{M}\chi \equiv 0$ by induction hypothesis,

$$\mathbf{F}0 \wedge (\mathbf{D}\psi \veebar \mathbf{T}\psi) \wedge \mathbf{M}\chi \equiv \mathbf{F}0 \wedge \mathbf{T}\psi \wedge 0 \equiv 0$$

by 4.4.2. Hence by (C3), $0\wedge\varphi \equiv (0\wedge\psi)\veebar(0\wedge\chi)$; so by another application of the induction hypothesis, $0\wedge\varphi \equiv 0$.

We now get $\mathbf{D}\varphi \equiv \mathbf{M}\varphi \equiv 0$ as before. ¶

Let $\mathbf{T}_4$ be the term algebra of signature $\{\veebar, \wedge, \downarrow, \bot, \top\}$ over an infinite set of variables, $\mathbf{T}_4/\mathcal{L}$ its quotient over $\mathcal{L}$-provable equality, $\mathbf{A}$ the reduct of $\mathbf{T}_4/\mathcal{L}$ to $\{\veebar, \wedge\}$, and $\mathbf{U}$ the subalgebra of $\mathbf{A}$ generated by the congruence classes of the test atoms.

4.7.5 Theorem. $\mathbf{U}$ is a distributive lattice. If we take $\veebar$ for join, then the least element of $\mathbf{U}$ is the congruence class of 0, and the greatest element the congruence class of 1.

Proof. We check the appropriate lattice laws, for test molecules φ, ψ, and χ. The associative laws are ($\veebar$1) and ($\wedge$1). By 4.7.4, $\uparrow\varphi \veebar \uparrow\psi \equiv 0$. Hence by (C1), $\varphi \veebar \psi \equiv 0 \veebar \varphi \veebar \psi \equiv 0 \veebar \psi \veebar \varphi \equiv \psi \veebar \varphi$. The commutative law for $\wedge$ is ($\wedge$2). Since $\uparrow\psi \equiv 0$ by 4.7.4, $\varphi \veebar (\varphi\wedge\psi) \equiv \varphi$ by (C4). The other absorption law follows by (C3). ($\veebar\wedge$) suffices for the distributive laws. The laws for 0 and 1 are in ($\veebar$3) and ($\wedge$5). ¶

4.7.6 Lemma. For test molecules φ, $\mathcal{L} \vdash \neg\varphi\equiv\mathbf{F}\varphi$.

Proof. By definition, $\neg\varphi=\mathbf{F}\varphi\veebar(\uparrow\varphi\wedge\varphi)$; and $\uparrow\varphi\wedge\varphi\equiv 0\wedge\varphi\equiv 0$ by 4.7.4.¶

4.7.7 Lemma. From $\mathcal{L}$ are derivable:

(a) $(\mathbf{x}\veebar\mathbf{y})\wedge\bot \equiv (\mathbf{x}\wedge\bot)\veebar(\mathbf{y}\wedge\bot)$; (b) $\mathbf{F}(\mathbf{x}\veebar\mathbf{y})\equiv\mathbf{F}\,\mathbf{x}\wedge\mathbf{F}\mathbf{y}$;

(c) $\downarrow\neg\mathbf{x} \equiv \downarrow\mathbf{x}$ (d) $\uparrow(0 \wedge \mathbf{x}) \equiv \uparrow(0 \wedge \mathbf{x}) \wedge \uparrow\mathbf{x}$

(e) $\mathbf{F}(\uparrow\mathbf{x} \wedge \mathbf{x}) \equiv \downarrow\mathbf{x}$ (f) $\mathbf{x} \veebar (\uparrow\mathbf{x} \wedge \mathbf{x}) \equiv \mathbf{x}$

(g) $\neg\neg\mathbf{x} \equiv \mathbf{x}$ (double negation).

Proof.
(a) Since $\mathbf{F}\bot \equiv 0$ (4.4.2), by 4.7.5 the substitution $\mathbf{x}\mapsto\bot$, $\mathbf{y}\mapsto\mathbf{x}$, $\mathbf{z}\mapsto\mathbf{y}$ makes the threshold of (C3) vanish.
(b) Hence $\mathbf{F}(\mathbf{x}\veebar\mathbf{y}) = \downarrow((\mathbf{x}\veebar\mathbf{y})\wedge\bot) \equiv \downarrow((\mathbf{x}\wedge\bot)\veebar(\mathbf{y}\wedge\bot))$. By 4.5.2(3), we get $\mathbf{Fx}\wedge(\mathbf{T}(\mathbf{x}\wedge\bot)\veebar\mathbf{Fy})$. Now apply (E3).
(c) By definition, $\downarrow\neg\mathbf{x} = \downarrow(\mathbf{Fx}\veebar(\uparrow\mathbf{x}\wedge\mathbf{x}))$. Apply 4.5.2(3) to obtain:

$$\downarrow\neg\mathbf{x} \equiv \downarrow\mathbf{Fx}\wedge(\mathbf{TFx}\veebar\downarrow(\uparrow\mathbf{x}\wedge\mathbf{x})).$$

By ($\downarrow$1), $\downarrow\mathbf{Fx}\equiv 1$. By 4.4.2, $\mathbf{TFx}\equiv\mathbf{Fx}$. By (D2),

$$\downarrow(\uparrow\mathbf{x}\wedge\mathbf{x}) \equiv (\mathbf{F}\uparrow\mathbf{x}\wedge\mathbf{Dx})\veebar(\mathbf{D}\uparrow\mathbf{x}\wedge\mathbf{Fx})\veebar(\downarrow\uparrow\mathbf{x}\wedge\downarrow\mathbf{x}).$$

By 4.4.2, $\mathbf{F}\uparrow\mathbf{x}\equiv\downarrow\mathbf{x}$. By definition, $\mathbf{Dx} = \uparrow\mathbf{x}\wedge\downarrow(0\wedge\mathbf{x})$. Applying ($\downarrow$3) and 4.4.2, we find that $\mathbf{F}\uparrow\mathbf{x}\wedge\mathbf{Dx}\equiv 0$. By 4.4.2(d), $\mathbf{D}\uparrow\mathbf{x}\equiv 0$, so $\mathbf{D}\uparrow\mathbf{x}\wedge\mathbf{Fx}\equiv 0$ by 4.4.2(b). Hence by ($\veebar$3), ($\downarrow$1), and ($\wedge$5),

$$\downarrow(\uparrow\mathbf{x}\wedge\mathbf{x})\equiv\downarrow\mathbf{x}.$$

We now have $\downarrow\neg\mathbf{x}\equiv\mathbf{Fx}\veebar\downarrow\mathbf{x}\equiv\downarrow\mathbf{x}$ by 4.5.2 and 4.7.5.
(d) By (D2),

$$\uparrow(0\wedge\mathbf{x}) = \mathbf{F}\downarrow(0\wedge\mathbf{x})\equiv\mathbf{F}[(\mathbf{F}0\wedge\mathbf{Dx})\veebar(\mathbf{D}0\wedge\mathbf{Fx})\veebar(\downarrow 0\wedge\downarrow\mathbf{x})].$$

By 4.4.2, the right-hand side reduces to $\mathbf{F}[\mathbf{Dx}\veebar\downarrow\mathbf{x}]$. By (b) we get

$$\uparrow(0\wedge\mathbf{x}) = \mathbf{FDx}\wedge\uparrow\mathbf{x},$$

and we are done by 4.2.
(e) By (D2),

$$\begin{aligned}\mathbf{F}(\uparrow\mathbf{x}\wedge\mathbf{x}) &\equiv \downarrow((\uparrow\mathbf{x}\wedge\bot)\wedge\mathbf{x})\\ &\equiv [\mathbf{F}(\uparrow\mathbf{x}\wedge\bot)\wedge\mathbf{Dx}]\veebar[\mathbf{D}(\uparrow\mathbf{x}\wedge\bot)\wedge\mathbf{Fx}]\veebar[\downarrow(\uparrow\mathbf{x}\wedge\bot)\wedge\downarrow\mathbf{x}].\end{aligned}$$

But $\mathbf{F}(\uparrow\mathbf{x}\wedge\bot) = \downarrow((\uparrow\mathbf{x}\wedge\bot)\wedge\bot)\equiv\mathbf{F}\uparrow\mathbf{x}\equiv\downarrow\mathbf{x}$ by 4.2 and 4.4, so by ($\downarrow$3),

$$\mathbf{F}(\uparrow\mathbf{x}\wedge\bot)\wedge\mathbf{Dx}\equiv 0.$$

By 4.4.3(a), $\mathbf{D}(\uparrow\mathbf{x}\wedge\bot)\equiv\uparrow\mathbf{x}$, so by 4.5.2(1), 4.7.5, and ($\downarrow$3),

$$\mathbf{D}(\uparrow\mathbf{x}\wedge\bot)\wedge\mathbf{Fx}\equiv 0.$$

Finally, $\downarrow(\uparrow\mathbf{x}\wedge\bot)\equiv\downarrow\mathbf{x}$ by 4.4.2(e).
(f) Since $\mathbf{M}\uparrow\mathbf{x}\equiv 0$ (4.4.2(d)), we may apply (C4), with $\uparrow\mathbf{x}$ for $\mathbf{y}$.
(g) By definition, $\neg\neg\mathbf{x} = \mathbf{Tx}\veebar(\uparrow\neg\mathbf{x}\wedge\neg\mathbf{x})$. By (c), $\uparrow\neg\mathbf{x}\equiv\uparrow\mathbf{x}$. Applying ($\veebar\wedge$) we get $\neg\neg\mathbf{x}\equiv(\mathbf{Tx}\veebar\uparrow\mathbf{x})\wedge(\mathbf{Tx}\veebar\neg\mathbf{x})$. By 4.5.2(1) and the definition of $\neg$, $\mathbf{Tx}\veebar\neg\mathbf{x} = \downarrow\mathbf{x}\veebar(\uparrow\mathbf{x}\wedge\mathbf{x})$. By 4.4.2(b),

$$\mathbf{M}(\uparrow\mathbf{x}\wedge\mathbf{x}) = \uparrow(\uparrow\mathbf{x}\wedge\mathbf{x}\wedge 0)\equiv\uparrow(\mathbf{x}\wedge 0) = \mathbf{Mx},$$

so by 4.4.2 and ($\downarrow$3) with (d),

$$(\mathbf{D}{\downarrow}\mathbf{x} \veebar \mathbf{T}{\downarrow}\mathbf{x}) \wedge \mathbf{M}({\uparrow}\mathbf{x} \wedge \mathbf{x}) \equiv {\downarrow}\mathbf{x} \wedge \mathbf{M}\mathbf{x} \equiv 0.$$

Hence we may apply (C3) to obtain

$$\neg\neg\mathbf{x} \equiv ((\mathbf{T}\mathbf{x} \veebar {\uparrow}\mathbf{x}) \wedge {\downarrow}\mathbf{x}) \veebar ((\mathbf{T}\mathbf{x} \veebar {\uparrow}\mathbf{x}) \wedge {\uparrow}\mathbf{x} \wedge \mathbf{x}).$$

By 4.7.5, 4.5.2(1), and ($\downarrow$3),

$$(\mathbf{T}\mathbf{x} \veebar {\uparrow}\mathbf{x}) \wedge {\downarrow}\mathbf{x} \equiv (\mathbf{T}\mathbf{x} \wedge {\downarrow}\mathbf{x}) \veebar ({\uparrow}\mathbf{x} \wedge {\downarrow}\mathbf{x}) \equiv \mathbf{T}\mathbf{x} \veebar 0 \equiv \mathbf{T}\mathbf{x}.$$

Again by 4.7.5, $(\mathbf{T}\mathbf{x} \veebar {\uparrow}\mathbf{x}) \wedge {\uparrow}\mathbf{x} \equiv {\uparrow}\mathbf{x}$. So $\neg\neg\mathbf{x} \equiv \mathbf{T}\mathbf{x} \veebar ({\uparrow}\mathbf{x} \wedge \mathbf{x})$.

By (b), $\mathbf{T}\mathbf{x} = \mathbf{F}\neg\mathbf{x} \equiv \mathbf{F}\mathbf{F}\mathbf{x} \wedge \mathbf{F}({\uparrow}\mathbf{x} \wedge \mathbf{x})$; so $\mathbf{T}\mathbf{x} \equiv \mathbf{F}\mathbf{F}\mathbf{x} \wedge {\downarrow}\mathbf{x}$ by (e). Since $\mathbf{D}\mathbf{F}\mathbf{F}\mathbf{x} \equiv \mathbf{D}{\downarrow}\mathbf{x} \equiv 0$ by 4.4.2(d), we may conclude by (C2) that

$$\neg\neg\mathbf{x} \equiv (\mathbf{F}\mathbf{F}\mathbf{x} \veebar ({\uparrow}\mathbf{x} \wedge \mathbf{x})) \wedge ({\downarrow}\mathbf{x} \veebar ({\uparrow}\mathbf{x} \wedge \mathbf{x})).$$

By ($\downarrow$4) and (f), $\mathbf{F}\mathbf{F}\mathbf{x} \veebar ({\uparrow}\mathbf{x} \wedge \mathbf{x}) \equiv {\uparrow}\mathbf{x} \veebar \mathbf{x}$. On the other hand, by ($\veebar\wedge$), 4.7.2(a), and ($\wedge$5), ${\downarrow}\mathbf{x} \veebar ({\uparrow}\mathbf{x} \wedge \mathbf{x}) \equiv {\downarrow}\mathbf{x} \veebar \mathbf{x}$. Since $\mathbf{D}{\downarrow}\mathbf{x} \equiv \mathbf{D}{\uparrow}\mathbf{x} \equiv 0$, we may apply (C2) (and ($\downarrow$3)) to conclude that

$$\neg\neg\mathbf{x} \equiv ({\uparrow}\mathbf{x} \veebar \mathbf{x}) \wedge ({\downarrow}\mathbf{x} \veebar \mathbf{x}) \equiv ({\uparrow}\mathbf{x} \wedge {\downarrow}\mathbf{x}) \veebar \mathbf{x} \equiv \mathbf{x}. \qquad ¶$$

Let us denote the congruence class under $\mathcal{L}$-provable equality of a term φ by $\varphi/\mathcal{L}$, and the operation $\varphi/\mathcal{L} \mapsto \neg\varphi/\mathcal{L}$ (which by 4.7.6 is the same as $\varphi/\mathcal{L} \mapsto \mathbf{F}\varphi/\mathcal{L}$) by $\neg$.

4.7.8 Theorem. $\langle \mathbf{U}, \neg, 0/\mathcal{L}, 1/\mathcal{L} \rangle$ is a Boolean algebra.

Proof. We must show that the laws of Boolean algebra hold; say, as stated in [K]. After 4.7.5 we need only check the complementation laws. So let φ be a test molecule. Then by 4.7.7(g) and 4.7.6, $\varphi \equiv \neg\neg\varphi \equiv \mathbf{F}\neg\varphi \equiv \mathbf{T}\varphi$, so by 4.7.6 again, 4.5.2(1), and 4.7.4, $\varphi \veebar \neg\varphi \equiv \mathbf{T}\varphi \veebar \mathbf{F}\varphi \equiv {\downarrow}\varphi \equiv 1$. On the other hand, by 4.7.7(b), 4.7.6, and 4.4.2(f), $\varphi \wedge \neg\varphi \equiv \mathbf{T}\varphi \wedge \mathbf{F}\varphi \equiv \mathbf{F}(\neg\varphi \veebar \varphi) \equiv \mathbf{F}1 \equiv 0$. ¶

Corollary. If φ is a test molecule, $\mathcal{L} \vdash \mathbf{T}\varphi \equiv \varphi$.

4.7.9 Lemma. (a) $\mathcal{L} \vdash {\uparrow}\mathbf{x} \veebar {\downarrow}\mathbf{x} \equiv 1$; (b) $\mathcal{L} \vdash \mathbf{M}\mathbf{x} \veebar \mathbf{D}\mathbf{x} \equiv {\uparrow}\mathbf{x}$.

Proof. (a) By 4.7.8, ${\uparrow}\mathbf{x} \veebar {\downarrow}\mathbf{x} \equiv \neg{\downarrow}\mathbf{x} \veebar {\downarrow}\mathbf{x} \equiv 1$.
(b) By 4.7.7(d), 4.7.5, and (a),

$$\begin{aligned}\mathbf{M}\mathbf{x} \veebar \mathbf{D}\mathbf{x} &= {\uparrow}(0 \wedge \mathbf{x}) \veebar ({\uparrow}\mathbf{x} \wedge {\downarrow}(0 \wedge \mathbf{x})) \\ &\equiv ({\uparrow}\mathbf{x} \wedge {\uparrow}(0 \wedge \mathbf{x})) \veebar ({\uparrow}\mathbf{x} \wedge {\downarrow}(0 \wedge \mathbf{x})) \\ &\equiv {\uparrow}\mathbf{x} \wedge ({\uparrow}(0 \wedge \mathbf{x}) \veebar {\downarrow}(0 \wedge \mathbf{x})) \equiv {\uparrow}\mathbf{x} \wedge 1 \equiv {\uparrow}\mathbf{x}. \qquad ¶\end{aligned}$$

4.7.10 Theorem. Let **x** be a variable. The atoms of the subalgebra $\mathbf{U}_{\mathbf{x}}$ of $\langle \mathbf{U}, \neg, 0/\mathcal{L}, 1/\mathcal{L} \rangle$ generated by $\{\kappa_a(\mathbf{x})/\mathcal{L} \mid a \in L\}$ are the elements $\kappa_a(\mathbf{x})/\mathcal{L}$ $(a \in L)$.

Proof. By the disjunctive normal form theorem of Boolean algebra, the elements of $\mathbf{U}_{\mathbf{x}}$ are represented by disjunctions of conjunctions of $\kappa_a(\mathbf{x})$ and their negations; by 4.7.6 we may write $\mathbf{F}\kappa_a(\mathbf{x})$ instead of $\neg\kappa_a(\mathbf{x})$. Since evidently $\kappa_a(\mathbf{x}) \equiv 0$ is not provable, it will suffice to show for each $a \in L$ that

$$\mathbf{F}\kappa_a(\mathbf{x}) \equiv \bigvee(\kappa_b(\mathbf{x}) \mid b \in L - \{a\}).$$

Recall that $\downarrow\mathbf{x} \equiv \mathbf{T}\mathbf{x} \vee \mathbf{F}\mathbf{x}$ (4.5.2(1)).
(i) $\mathbf{F}\mathbf{D}\mathbf{x} \equiv \downarrow\mathbf{x} \vee \mathbf{M}\mathbf{x}$: By 4.7.8, $\mathbf{F}\mathbf{D}\mathbf{x} = \mathbf{F}(\uparrow\mathbf{x} \wedge \downarrow(0 \wedge \mathbf{x})) \equiv \mathbf{F}\uparrow\mathbf{x} \vee \uparrow(0 \wedge \mathbf{x})$. But $\mathbf{F}\uparrow\mathbf{x} \equiv \downarrow\mathbf{x}$ by 4.4.2, and $\uparrow(0 \wedge \mathbf{x}) = \mathbf{M}\mathbf{x}$.
(ii) $\mathbf{F}\mathbf{M}\mathbf{x} \equiv \mathbf{D}\mathbf{x} \vee \downarrow\mathbf{x}$: By 4.4.2, (D2), and 4.7.5,

$$\begin{aligned}\mathbf{F}\mathbf{M}\mathbf{x} &\equiv \downarrow(0 \wedge \mathbf{x}) \equiv (\mathbf{F}0 \wedge \mathbf{D}\mathbf{x}) \vee (\mathbf{D}0 \wedge \mathbf{F}\mathbf{x}) \vee (\downarrow 0 \wedge \downarrow\mathbf{x}) \\ &\equiv (1 \wedge \mathbf{D}\mathbf{x}) \vee (0 \wedge \mathbf{F}\mathbf{x}) \vee (1 \wedge \downarrow\mathbf{x}) \equiv \mathbf{D}\mathbf{x} \vee \downarrow\mathbf{x}.\end{aligned}$$

(iii) $\mathbf{F}\mathbf{F}\mathbf{x} \equiv \mathbf{T}\mathbf{x} \vee \uparrow\mathbf{x}$: By 4.7.8 and 4.4.2,

$$\mathbf{F}(\mathbf{T}\mathbf{x} \vee \uparrow\mathbf{x}) \equiv \mathbf{F}\mathbf{T}\mathbf{x} \wedge \mathbf{F}\uparrow\mathbf{x} \equiv \mathbf{F}\mathbf{T}\mathbf{x} \wedge \downarrow\mathbf{x} \equiv \mathbf{F}\mathbf{T}\mathbf{x} \wedge \mathbf{F}\mathbf{x}.$$

By 4.7.7(b) and 4.7.8,

$$\mathbf{F}\mathbf{T}\mathbf{x} = \mathbf{F}\mathbf{F}\neg\mathbf{x} = \mathbf{F}\mathbf{F}(\mathbf{F}\mathbf{x} \vee (\uparrow\mathbf{x} \wedge \mathbf{x})) \equiv \mathbf{F}\mathbf{x} \vee \mathbf{F}\mathbf{F}(\uparrow\mathbf{x} \wedge \mathbf{x});$$

by absorption $\mathbf{F}(\mathbf{T}\mathbf{x} \vee \uparrow\mathbf{x}) \equiv \mathbf{F}\mathbf{x}$. Hence by 4.7.8 $\mathbf{T}\mathbf{x} \vee \uparrow\mathbf{x} \equiv \mathbf{F}\mathbf{F}\mathbf{x}$.
(iv) $\mathbf{F}\mathbf{T}\mathbf{x} \equiv \mathbf{F}\mathbf{x} \vee \uparrow\mathbf{x}$: By (iii), $\mathbf{F}\mathbf{T}\mathbf{x} \equiv \mathbf{F}\mathbf{F}\neg\mathbf{x} \equiv \mathbf{T}\neg\mathbf{x} \vee \uparrow\neg\mathbf{x}$. Now observe that $\mathbf{T}\neg\mathbf{x} = \mathbf{F}\neg\neg\mathbf{x} \equiv \mathbf{F}\mathbf{x}$, and $\uparrow\neg\mathbf{x} \equiv \uparrow\mathbf{x}$, by 4.7.7. ¶

4.7.11 Lemma. $\mathcal{L} \vdash \mathbf{F}\mathbf{D}\mathbf{x} \vee \mathbf{x} \equiv \mathbf{F}\mathbf{D}\mathbf{x} \vee \bot$, $\mathbf{F}\mathbf{M}\mathbf{x} \vee \mathbf{x} \equiv \mathbf{F}\mathbf{M}\mathbf{x} \vee \top$.

Proof. The first equation is immediate by 4.7.10 and (C5). The second follows from (C6) since by 4.4, $\mathbf{F}\uparrow(0 \wedge \mathbf{x}) \equiv \downarrow(0 \wedge \mathbf{x})$. ¶

4.7.12 Lemma. Suppose φ is a test molecule, and **c** is either $\bot$ or $\top$. Then the following equations are derivable from $\mathcal{L}$:
(a) $\downarrow(\varphi \vee \mathbf{c}) \equiv \mathbf{T}(\varphi \vee \mathbf{c}) \equiv \varphi$ and $\mathbf{D}(\varphi \vee \top) \equiv \mathbf{M}(\varphi \vee \bot) \equiv \mathbf{F}(\varphi \vee \mathbf{c}) \equiv 0$;
(b) $\neg(\varphi \vee \bot) \equiv \mathbf{F}\varphi \wedge \bot$;
(c) $\mathbf{D}(\varphi \vee \bot) \equiv \mathbf{M}(\varphi \vee \top) \equiv \mathbf{F}\varphi$.

Proof. Let **c** be $\bot$ or $\top$.
(a), first instalment. By 4.5.2(3), 4.7.4, 4.7.8, 4.4.2(a), and 4.1, 2,

$$\downarrow(\varphi \veebar \mathbf{c}) \equiv \downarrow\varphi \wedge (\mathbf{T}\varphi \veebar \downarrow\mathbf{c}) \equiv 1 \wedge (\varphi \veebar 0) \equiv \varphi.$$

$\mathbf{F}(\varphi \veebar \mathbf{c}) \equiv 0$ by 4.7.7(b) and 4.4.2. By (E2) and 4.7.4, $\mathbf{D}(\varphi \veebar \top) \equiv \mathbf{D}\varphi \equiv 0$. Since $\mathbf{M}\bot \equiv 0$, we may use (C3) to conclude

$$\mathbf{M}(\varphi \veebar \bot) \equiv \uparrow((0 \wedge \varphi) \veebar (0 \wedge \bot));$$

hence by 4.7.4 and 4.4.2, $\mathbf{M}(\varphi \veebar \bot) \equiv \uparrow 0 \equiv 0$.

(b) Using (a) we find

$$\begin{aligned}\neg(\varphi \veebar \bot) &= \mathbf{F}(\varphi \veebar \bot) \veebar (\uparrow(\varphi \veebar \bot) \wedge (\varphi \veebar \bot)) \\ &\equiv 0 \veebar (\mathbf{F}\varphi \wedge (\varphi \veebar \bot)) \\ &\equiv (\mathbf{F}\varphi \wedge \varphi) \veebar (\mathbf{F}\varphi \wedge \bot) \qquad \text{by 4.1, 4.4.2 and (C3)} \\ &\equiv \mathbf{F}\varphi \wedge \bot \qquad \text{by 4.7.8 and 4.1.}\end{aligned}$$

(a), remainder. $\mathbf{T}(\varphi \veebar \top) \equiv \varphi$ is immediate by (E1) and 4.7.8. On the other hand, by (b), 4.2, and 4.7.8, $\mathbf{T}(\varphi \veebar \bot) = \mathbf{F}\neg(\varphi \veebar \bot) \equiv \mathbf{F}(\mathbf{F}\varphi \wedge \bot) \equiv \mathbf{F}\mathbf{F}\varphi \equiv \varphi$.

(c) By definition, $\mathbf{D}(\varphi \veebar \bot) = \uparrow(\varphi \veebar \bot) \wedge \downarrow(0 \wedge (\varphi \veebar \bot))$. By (a),

$$\uparrow(\varphi \veebar \bot) \equiv \mathbf{F}\downarrow(\varphi \veebar \bot) \equiv \mathbf{F}\varphi.$$

As for the other conjunct, since $\mathbf{M}\bot \equiv 0$ (4.4.2), we may apply (C3) to obtain $\downarrow((0 \wedge \varphi) \veebar (0 \wedge \bot))$; by 4.4.2, 4.7.4 and 4.1, this reduces to 1; so this part vanishes by 4.2.

By (D2), 4.4.2, 4.1, 2, and (a),

$$\begin{aligned}\mathbf{M}(\varphi \veebar \top) &= \mathbf{F}\downarrow(0 \wedge (\varphi \veebar \top)) \\ &\equiv \mathbf{F}[(\mathbf{F}0 \wedge \mathbf{D}(\varphi \veebar \top)) \veebar (\mathbf{D}0 \wedge \mathbf{F}(\varphi \veebar \top)) \veebar (\downarrow 0 \wedge \downarrow(\varphi \veebar \top))] \\ &\equiv \mathbf{F}\varphi. \qquad ¶\end{aligned}$$

4.7.13 Lemma. The following equations are derivable from $\mathcal{L}$:

(a) $\mathbf{D}(\mathbf{x} \wedge \bot) \equiv \mathbf{T}\mathbf{x} \veebar \mathbf{D}\mathbf{x}$

(b) $\mathbf{F}(\mathbf{x} \wedge \mathbf{y}) \equiv (\mathbf{F}\mathbf{x} \wedge \mathbf{T}\mathbf{y}) \veebar (\mathbf{F}\mathbf{x} \wedge \mathbf{D}\mathbf{y}) \veebar (\mathbf{D}\mathbf{x} \wedge \mathbf{F}\mathbf{y}) \veebar (\downarrow\mathbf{x} \wedge \mathbf{F}\mathbf{y})$

(c) $\mathbf{D}(\mathbf{x} \wedge \mathbf{y}) \equiv (\mathbf{D}\mathbf{x} \wedge \mathbf{D}\mathbf{y}) \veebar (\mathbf{T}\mathbf{x} \wedge \mathbf{D}\mathbf{y}) \veebar (\mathbf{D}\mathbf{x} \wedge \mathbf{T}\mathbf{y})$

(d) $\mathbf{M}(\mathbf{x} \wedge \mathbf{y}) \equiv \mathbf{M}\mathbf{x} \veebar \mathbf{M}\mathbf{y}$

(e) $\mathbf{T}(\mathbf{x} \wedge \mathbf{y}) \equiv \mathbf{T}\mathbf{x} \wedge \mathbf{T}\mathbf{y}$

Proof.

(a) By definition, $\mathbf{D}(\mathbf{x} \wedge \bot) = \uparrow(\mathbf{x} \wedge \bot) \wedge \downarrow(0 \wedge \mathbf{x} \wedge \bot)$. The first conjunct is $\mathbf{F}\mathbf{F}\mathbf{x}$; the second, by 4.4.2, is equivalent to $\mathbf{F}\mathbf{M}\mathbf{x}$. Now use 4.7.10.

(b) By (D2),

$$\begin{aligned}\mathbf{F}(\mathbf{x} \wedge \mathbf{y}) &= \downarrow(\mathbf{x} \wedge \mathbf{y} \wedge \bot) \\ &\equiv (\mathbf{F}\mathbf{x} \wedge \mathbf{D}(\mathbf{y} \wedge \bot)) \veebar (\mathbf{D}\mathbf{x} \wedge \mathbf{F}(\mathbf{y} \wedge \bot)) \veebar (\downarrow\mathbf{x} \wedge \downarrow(\mathbf{y} \wedge \bot)).\end{aligned}$$

By (a) and 4.7.5, $\mathbf{Fx} \wedge \mathbf{D}(\mathbf{y} \wedge \bot) \equiv (\mathbf{Fx} \wedge \mathbf{Ty}) \veebar (\mathbf{Fx} \wedge \mathbf{Dy})$; and by $(\wedge)$, $\mathbf{F}(\mathbf{y} \wedge \bot) \equiv \mathbf{Fy}$.
(c) By definition $\mathbf{D}(\mathbf{x} \wedge \mathbf{y}) = \uparrow(\mathbf{x} \wedge \mathbf{y}) \wedge \downarrow(0 \wedge \mathbf{x} \wedge \mathbf{y})$. By (D2),

$$\uparrow(\mathbf{x} \wedge \mathbf{y}) = \mathbf{F}\downarrow(\mathbf{x} \wedge \mathbf{y}) \equiv \mathbf{F}((\mathbf{Fx} \wedge \mathbf{Dy}) \veebar (\mathbf{Dx} \wedge \mathbf{Fy}) \veebar (\downarrow\mathbf{x} \wedge \downarrow\mathbf{y})).$$

By 4.7.8, 10 the right-hand side reduces to

$$(*) \qquad \mathbf{Mx} \veebar (\mathbf{FFx} \wedge \uparrow\mathbf{y}) \veebar (\uparrow\mathbf{x} \wedge \mathbf{FFy}) \veebar \mathbf{My}.$$

On the other hand, by (D2),

$$\downarrow(0 \wedge \mathbf{x} \wedge \mathbf{y}) \equiv (\mathbf{F}(0 \wedge \mathbf{x}) \wedge \mathbf{Dy}) \veebar (\mathbf{D}(0 \wedge \mathbf{x}) \wedge \mathbf{Fy}) \veebar (\downarrow(0 \wedge \mathbf{x}) \wedge \downarrow\mathbf{y}).$$

But $\mathbf{D}(0 \wedge \mathbf{x}) \equiv 0$ by $(\downarrow 3)$; and $\mathbf{F}(0 \wedge \mathbf{x}) \equiv \downarrow(0 \wedge \mathbf{x})$ by 4.4.2(b); so

$$(\dagger) \qquad \downarrow(0 \wedge \mathbf{x} \wedge \mathbf{y}) \equiv (\downarrow(0 \wedge \mathbf{x}) \wedge \mathbf{Dy}) \veebar (\downarrow(0 \wedge \mathbf{x}) \wedge \downarrow\mathbf{y}).$$

Now use 4.7.8, 10 to combine (*) and (†).
(d) We begin by noting that $\mathbf{D}(\mathbf{y} \wedge 0) = \uparrow(\mathbf{y} \wedge 0) \wedge \downarrow(\mathbf{y} \wedge 0 \wedge 0) \equiv 0$ (4.2, $\downarrow 3$) and $\mathbf{F}(\mathbf{y} \wedge 0) \equiv \downarrow(\mathbf{y} \wedge 0 \wedge \bot) \equiv \mathbf{FMy}$ (4.4.2). Hence by (D2), 4.7.10, 8

$$\begin{aligned}\mathbf{M}(\mathbf{x} \wedge \mathbf{y}) &= \mathbf{F}\downarrow(\mathbf{x} \wedge \mathbf{y} \wedge 0)\\ &\equiv \mathbf{F}((\mathbf{Fx} \wedge \mathbf{D}(\mathbf{y} \wedge 0)) \veebar (\mathbf{Dx} \wedge \mathbf{F}(\mathbf{y} \wedge 0)) \veebar (\downarrow\mathbf{x} \wedge \downarrow(\mathbf{y} \wedge 0))\\ &\equiv \mathbf{F}((\mathbf{Dx} \wedge \mathbf{FMy}) \veebar (\downarrow\mathbf{x} \wedge \mathbf{FMy})) \equiv \mathbf{F}(\mathbf{FMx} \wedge \mathbf{FMy})\\ &\equiv \mathbf{Mx} \veebar \mathbf{My}.\end{aligned}$$

(e) By 4.7.10, 8 and (b)

$$\begin{aligned}\mathbf{T}(\mathbf{x} \wedge \mathbf{y}) &\equiv \mathbf{F}(\uparrow(\mathbf{x} \wedge \mathbf{y}) \veebar \mathbf{F}(\mathbf{x} \wedge \mathbf{y})) \equiv \downarrow(\mathbf{x} \wedge \mathbf{y}) \wedge \mathbf{FF}(\mathbf{x} \wedge \mathbf{y}))\\ &\equiv [(\mathbf{Fx} \wedge \mathbf{Dy}) \veebar (\mathbf{Dx} \wedge \mathbf{Fy}) \veebar (\downarrow\mathbf{x} \wedge \downarrow\mathbf{y})]\\ &\qquad \wedge (\mathbf{FFx} \veebar \mathbf{FTy}) \wedge (\mathbf{FFx} \veebar \mathbf{FDy})\\ &\qquad \wedge (\mathbf{FDx} \veebar \mathbf{FFy}) \wedge (\uparrow\mathbf{x} \veebar \mathbf{FFy}).\end{aligned}$$

But $(\mathbf{FFx} \veebar \mathbf{FTy}) \wedge (\mathbf{FFx} \veebar \mathbf{FDy}) \equiv \mathbf{FFx} \veebar \mathbf{Fy} \veebar \mathbf{My}$ and

$$(\mathbf{FDx} \veebar \mathbf{FFy}) \wedge (\uparrow\mathbf{x} \veebar \mathbf{FFy}) \equiv \mathbf{Mx} \veebar \mathbf{FFy};$$

whence distributing in two steps, and using 4.7.10, we find

$$\begin{aligned}\mathbf{T}(\mathbf{x} \wedge \mathbf{y}) &\equiv [(\mathbf{Dx} \wedge \mathbf{Fy}) \veebar (\mathbf{Tx} \wedge \downarrow\mathbf{y}) \veebar (\downarrow\mathbf{x} \wedge \mathbf{Fy})] \wedge (\mathbf{Mx} \veebar \mathbf{FFy})\\ &\equiv \mathbf{Tx} \wedge \mathbf{Ty}. \qquad ¶\end{aligned}$$

4.7.14 Lemma. $\mathcal{L} \vdash \mathbf{x} \veebar \neg\mathbf{x} \equiv \mathbf{x} \veebar \mathbf{Fx}$, $\mathbf{FTx} \veebar \mathbf{x} \equiv 1$.

Proof. $\mathbf{x} \veebar \neg\mathbf{x} \equiv \mathbf{x} \veebar \mathbf{Fx}$: By 4.7.7(f), $\mathbf{x} \veebar (\uparrow\mathbf{x} \wedge \mathbf{x}) \equiv \mathbf{x}$. Use the definition of $\neg$ and 4.1.

$\mathbf{FTx} \curlyvee \mathbf{x} \equiv 1$: Substituting $\neg\mathbf{x}$ for $\mathbf{x}$ in the equation we just proved, and applying double negation, we get $\neg\mathbf{x} \curlyvee \mathbf{x} \equiv \neg\mathbf{x} \curlyvee \mathbf{Tx}$. By ($\downarrow$4), $\mathbf{FTx} \equiv \uparrow\neg\mathbf{x} \curlyvee \neg\mathbf{x}$. Hence $\mathbf{FTx} \curlyvee \mathbf{x} \equiv \uparrow\neg\mathbf{x} \curlyvee \neg\mathbf{x} \curlyvee \mathbf{Tx} \equiv \mathbf{FTx} \curlyvee \mathbf{Tx} \equiv 1$ by 4.7.8. ¶

5. Normal forms

For $a \in L$ we define

$$\nu_a\mathbf{x} = \mathbf{F}\kappa_a\mathbf{x}$$

A *criterion* for a set $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ ($\mathbf{x}_i \neq \mathbf{x}_j$ for $i \neq j$) of variables is a conjunction

$$(*) \qquad \bigwedge_{i=1}^{n} \kappa_{a_i}\mathbf{x}_i \quad ,$$

a *dual criterion* a left-first disjunction

$$(\dagger) \qquad \mathop{\curlyvee}_{i=1}^{n} \nu_{a_i}\mathbf{x}_i \quad ,$$

where $a_1,\ldots, a_n$ are elements of L. We shall often use the notations

$$\kappa_{\langle a_1,\ldots,a_n\rangle}(\mathbf{x}_1,\ldots,\mathbf{x}_n), \qquad \nu_{\langle a_1,\ldots,a_n\rangle}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$$

for (*) and (†); in fact, most of the time we write $\kappa_{\boldsymbol{a}}$ and $\nu_{\boldsymbol{a}}$ ($\boldsymbol{a} \in L^n$), assuming $\mathbf{x}_1, \ldots, \mathbf{x}_n$ fixed. By 4.7.8, $\mathbf{F}\nu_{\boldsymbol{a}} \equiv \kappa_{\boldsymbol{a}}$ and $\mathbf{F}\kappa_{\boldsymbol{a}} \equiv \nu_{\boldsymbol{a}}$. Moreover, for distinct $\boldsymbol{a}, \boldsymbol{b} \in L^n$, $\kappa_{\boldsymbol{a}} \wedge \kappa_{\boldsymbol{b}} \equiv 0$ by 4.7.10, hence $\kappa_{\boldsymbol{a}} \wedge \nu_{\boldsymbol{b}} \equiv \kappa_{\boldsymbol{a}}$ and $\nu_{\boldsymbol{a}} \curlyvee \nu_{\boldsymbol{b}} \equiv 1$, and $\nu_{\boldsymbol{a}} \curlyvee \kappa_{\boldsymbol{b}} \equiv \nu_{\boldsymbol{a}}$.

A *clause* for $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ is a term of the form

$$(\ddagger) \qquad \nu_{\langle a_1,\ldots,a_n\rangle}(\mathbf{x}_1,\ldots,\mathbf{x}_n) \curlyvee \mathbf{c},$$

with $\mathbf{c} \in \{\bot,\top,0,1\}$. A *reduced clause* is either a dual criterion or a clause (‡) with $\mathbf{c} \in \{\bot,\top\}$. Two clauses $\nu_{\boldsymbol{a}} \curlyvee \mathbf{c}$ and $\nu_{\boldsymbol{b}} \curlyvee \mathbf{d}$, or a dual criterion $\nu_{\boldsymbol{a}}$ and a clause $\nu_{\boldsymbol{b}} \curlyvee \mathbf{d}$, or dual criteria $\nu_{\boldsymbol{a}}$ and $\nu_{\boldsymbol{b}}$, are *independent* if $\boldsymbol{a} \neq \boldsymbol{b}$.

Definition. Let X be a finite set of variables. An X-*normal form* is a parallel conjunction of pairwise independent reduced clauses for X. A term φ *has an* X-*normal form* if φ is provably equal to an X-normal form.

Normal Form Theorem. Every term in variables from X has an X-normal form.

The proof occupies the rest of this section.

5.1 Lemma. $\mathcal{L} \vdash \bigwedge_{\boldsymbol{a} \in L^n} \nu_{\boldsymbol{a}}(\mathbf{x}_1,\ldots,\mathbf{x}_n) \equiv 0$.

Proof. By induction on n.
$n = 0$. In this case the empty sequence $\varnothing$ is the only element of L^n, and $\nu_\varnothing$ is the empty left-first disjunction 0.
$n = m+1$. We note that $\bigwedge_{a \in L} \nu_a(\mathbf{x}) \equiv 0$ by 4.7.10. Now we calculate:

$$\begin{aligned}
\bigwedge_{\boldsymbol{a} \in L^n} \nu_{\boldsymbol{a}}(\mathbf{x}_1,\ldots,\mathbf{x}_n) &= \bigwedge_{\boldsymbol{a} \in L^m} \bigwedge_{b \in L} (\nu_{\boldsymbol{a}}(\mathbf{x}_1,\ldots,\mathbf{x}_m) \curlyvee \nu_b(\mathbf{x}_n)) \\
&\equiv \bigwedge_{\boldsymbol{a} \in L^m} (\nu_{\boldsymbol{a}}(\mathbf{x}_1,\ldots,\mathbf{x}_m) \curlyvee \bigwedge_{b \in L} \nu_b(\mathbf{x}_n)), \text{ by } (\curlyvee\wedge) \\
&\equiv \bigwedge_{\boldsymbol{a} \in L^m} (\nu_{\boldsymbol{a}}(\mathbf{x}_1,\ldots,\mathbf{x}_m) \curlyvee 0) \\
&\equiv \bigwedge_{\boldsymbol{a} \in L^m} \nu_{\boldsymbol{a}}(\mathbf{x}_1,\ldots,\mathbf{x}_m) \equiv 0, \text{ by induction hypothesis. } ¶
\end{aligned}$$

5.2 Lemma. Let $\mathbf{y}$ be some variable. Then

$$\mathcal{L} \vdash \mathbf{y} \equiv \bigwedge_{\boldsymbol{a} \in L^n} (\nu_{\boldsymbol{a}}(\mathbf{x}_1,\ldots,\mathbf{x}_n) \curlyvee \mathbf{y}).$$

Proof. By ($\curlyvee$3) and 5.1,

$$\mathbf{y} \equiv 0 \curlyvee \mathbf{y} \equiv \bigwedge_{\boldsymbol{a} \in L^n} \nu_{\boldsymbol{a}}(\mathbf{x}_1,\ldots,\mathbf{x}_n) \curlyvee \mathbf{y}.$$

By 4.7.4, $\mathbf{D} \bigwedge_{\boldsymbol{a} \in A} \nu_{\boldsymbol{a}}(\mathbf{x}_1,\ldots,\mathbf{x}_n) \equiv 0$ for every nonvoid $A \subseteq L^n$. Hence by (C2) we may distribute $\mathbf{y}$ into the conjunction. ¶

For the sequel we set $X = \{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$, and where possible write $\kappa_{\boldsymbol{a}}$, $\nu_{\boldsymbol{a}}$ instead of $\kappa_{\boldsymbol{a}}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$, $\nu_{\boldsymbol{a}}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$.

By substitition for $\mathbf{y}$ in the identity of lemma 5.2, we see that $\bot$ and $\top$ have X-normal forms. By 5.1, $\bigwedge_{\boldsymbol{a} \in L^n} \nu_{\boldsymbol{a}}$ is an X-normal form of 0. Since 1 *is* a normal form, we have

Corollary A. Each constant has an X-normal form.

Moreover,

Corollary B. Let $1 \le i \le n$. Then $\mathbf{x}_i$ has an X-normal form.

Proof. We have

$$\mathbf{x}_i \equiv \bigwedge_{\boldsymbol{a} \in L^n} (\nu_{\boldsymbol{a}} \curlyvee \mathbf{x}_i)$$

by 5.2. For each dual criterion ν_a, consider the disjunct that refers to $\mathbf{x}_i$. If it is $\mathbf{FM}\mathbf{x}_i$ or $\mathbf{FD}\mathbf{x}_i$, then by 4.1 and 4.7.11(a), $\nu_a \curlyvee \mathbf{x}_i \equiv \nu_a \curlyvee \mathbf{c}$, where $\mathbf{c} = \mathsf{T}, \bot$ respectively. If it is $\mathbf{FF}\mathbf{x}_i$, then by 4.1 and 4.4.3(b), $\nu_a \curlyvee \mathbf{x}_i \equiv \nu_a$. Finally, if it is $\mathbf{FT}\mathbf{x}_i$, then by 4.1, 4.7.8, 4.7.14, $\nu_a \curlyvee \mathbf{x}_i \equiv 1$. ¶

5.3 Lemma. Let $\varphi =$

$$\bigwedge_{a \in A} \nu_a \wedge \bigwedge_{b \in B} (\nu_b \curlyvee \bot) \wedge \bigwedge_{c \in C} (\nu_c \curlyvee \mathsf{T}),$$

be an X-normal form, with $A, B, C \subseteq L^n$. Then

$$\mathbf{F}\varphi \equiv \bigwedge(\nu_u \mid u \in L^n - A), \qquad \mathbf{T}\varphi \equiv \bigwedge(\nu_u \mid u \in A \cup B \cup C),$$

$$\mathbf{D}\varphi \equiv \bigwedge(\nu_u \mid u \in L^n - B), \qquad \mathbf{M}\varphi \equiv \bigwedge(\nu_u \mid u \in L^n - C),$$

$$\downarrow\varphi \equiv \bigwedge(\nu_u \mid u \in B \cup C), \qquad \uparrow\varphi \equiv \bigwedge(\nu_u \mid u \in L^n - (B \cup C)).$$

Proof. By induction on the cardinality of $A \cup B \cup C$. The last two formulas follow from the first four by 4.7.8 and 4.5.2(1) and 4.7.9.
Basis: $A \cup B \cup C = \varnothing$. Use 4.4.2 and 5.1.
Induction step: we must take into account the possibility that one or two of A, B, C are void. Let φ be as above; we consider normal forms

(i) $\varphi \wedge \nu_d$,

(ii) $\varphi \wedge (\nu_e \curlyvee \bot)$,

(iii) $\varphi \wedge (\nu_f \curlyvee \mathsf{T})$

(i) By 4.7.13(b), 4.7.8, and 4.7.4,

$$\begin{aligned}\mathbf{F}(\varphi \wedge \nu_d) &\equiv (\mathbf{F}\varphi \wedge \mathbf{T}\nu_d) \curlyvee (\mathbf{F}\varphi \wedge \mathbf{D}\nu_d) \curlyvee (\mathbf{D}\varphi \wedge \mathbf{F}\nu_d) \curlyvee (\downarrow\varphi \wedge \mathbf{F}\nu_d) \\ &\equiv (\mathbf{F}\varphi \wedge \nu_d) \curlyvee (\mathbf{D}\varphi \wedge \kappa_d) \curlyvee (\downarrow\varphi \wedge \kappa_d).\end{aligned}$$

By induction hypothesis, the first disjunct reduces to

$$\bigwedge(\nu_u \mid u \in L^n - A) \wedge \nu_d \equiv \bigwedge(\nu_u \mid u \in L^n - A) \equiv \mathbf{F}\varphi$$

since $d \notin A$. Since $d \notin B$, the second disjunct vanishes. So we have

$$\begin{aligned}\mathbf{F}(\varphi \wedge \nu_d) &\equiv \mathbf{F}\varphi \curlyvee (\downarrow\varphi \wedge \kappa_d) \equiv (\mathbf{F}\varphi \curlyvee \downarrow\varphi) \wedge (\mathbf{F}\varphi \curlyvee \kappa_d) \\ &\equiv \downarrow\varphi \wedge (\mathbf{F}\varphi \curlyvee \kappa_d),\end{aligned}$$

by 4.7.8 and 4.5.2. Now distribute κ_d into a normal form of $\mathbf{F}\varphi$. Recall that if $u \neq d$, then $\nu_u \curlyvee \kappa_d \equiv \nu_u$, whereas $\nu_d \curlyvee \kappa_d \equiv 1$; so the result is equivalent to

$$\bigwedge(\nu_u \mid u \in L^n - (A \cup \{d\})),$$

and this absorbs $\downarrow\varphi$.

By 4.7.13(e) and 4.7.8 and the induction hypothesis,

$$\mathbf{T}(\varphi \wedge \nu_{\boldsymbol{d}}) \equiv \mathbf{T}\varphi \wedge \mathbf{T}\nu_{\boldsymbol{d}} \equiv \bigwedge(\nu_{\boldsymbol{u}} \mid \boldsymbol{u} \in A \cup \{\boldsymbol{d}\} \cup B \cup C).$$

By 4.7.13(c),

$$\mathbf{D}(\varphi \wedge \nu_{\boldsymbol{d}}) \equiv (\mathbf{D}\varphi \wedge \mathbf{D}\nu_{\boldsymbol{d}}) \veebar (\mathbf{T}\varphi \wedge \mathbf{D}\nu_{\boldsymbol{d}}) \veebar (\mathbf{D}\varphi \wedge \mathbf{T}\nu_{\boldsymbol{d}}).$$

But by 4.7.4, 5, the first two disjuncts vanish; so we are done by induction hypothesis and 4.7.8.

Finally, by 4.7.13(d),

$$\mathbf{M}(\varphi \wedge \nu_{\boldsymbol{d}}) \equiv \mathbf{M}\varphi \veebar \mathbf{M}\nu_{\boldsymbol{d}} \equiv \mathbf{M}\varphi \veebar 0 \equiv \mathbf{M}\varphi.$$

(ii) By 4.7.12, $\mathbf{T}(\nu_{\boldsymbol{e}} \veebar \bot) \equiv \nu_{\boldsymbol{e}}$, $\mathbf{D}(\nu_{\boldsymbol{e}} \veebar \bot) \equiv \mathbf{F}\nu_{\boldsymbol{e}}$, and $\mathbf{F}(\nu_{\boldsymbol{e}} \veebar \bot) \equiv 0$; so by 4.7.13(b) and 4.7.8,

$$\mathbf{F}(\varphi \wedge (\nu_{\boldsymbol{e}} \veebar \bot)) \equiv (\mathbf{F}\varphi \wedge \nu_{\boldsymbol{e}}) \veebar (\mathbf{F}\varphi \wedge \mathbf{F}\nu_{\boldsymbol{e}}) \equiv \mathbf{F}\varphi.$$

By 4.7.13(e) and the induction hypothesis,

$$\mathbf{T}(\varphi \wedge (\nu_{\boldsymbol{e}} \veebar \bot)) \equiv \mathbf{T}\varphi \wedge \nu_{\boldsymbol{e}} \equiv \bigwedge(\nu_{\boldsymbol{u}} \mid \boldsymbol{u} \in A \cup B \cup \{\boldsymbol{e}\} \cup C).$$

By 4.7.13(c) and 4.7.8,

$$\begin{aligned}\mathbf{D}(\varphi \wedge (\nu_{\boldsymbol{e}} \veebar \bot)) &\equiv (\mathbf{D}\varphi \wedge \mathbf{F}\nu_{\boldsymbol{e}}) \veebar (\mathbf{T}\varphi \wedge \mathbf{F}\nu_{\boldsymbol{e}}) \veebar (\mathbf{D}\varphi \wedge \nu_{\boldsymbol{e}}) \\ &\equiv \mathbf{D}\varphi \veebar (\mathbf{T}\varphi \wedge \kappa_{\boldsymbol{e}}).\end{aligned}$$

Since $\boldsymbol{e} \notin A \cup B \cup C$, $\mathbf{T}\varphi \wedge \kappa_{\boldsymbol{e}} \equiv \kappa_{\boldsymbol{e}}$. Now distribute $\kappa_{\boldsymbol{e}}$ into a normal form of $\mathbf{D}\varphi$ as given by induction hypothesis, and recall that $\nu_{\boldsymbol{u}} \veebar \kappa_{\boldsymbol{e}} \equiv \nu_{\boldsymbol{u}}$ if $\boldsymbol{u} \neq \boldsymbol{e}$, whereas $\nu_{\boldsymbol{e}} \veebar \kappa_{\boldsymbol{e}} \equiv 1$.

By 4.7.12, $\mathbf{M}(\nu_{\boldsymbol{e}} \veebar \bot) \equiv 0$, so by 4.7.13(d),

$$\mathbf{M}(\varphi \wedge (\nu_{\boldsymbol{e}} \veebar \bot)) \equiv \mathbf{M}\varphi \veebar \mathbf{M}(\nu_{\boldsymbol{e}} \veebar \bot) \equiv \mathbf{M}\varphi.$$

(iii) By 4.7.12, $\mathbf{T}(\nu_{\boldsymbol{f}} \veebar \top) \equiv \nu_{\boldsymbol{f}}$ and $\mathbf{D}(\nu_{\boldsymbol{f}} \veebar \top) \equiv \mathbf{F}(\nu_{\boldsymbol{f}} \veebar \top) \equiv 0$; so by 4.7.13(b) and since $\boldsymbol{f} \notin A$, $\mathbf{F}(\varphi \wedge (\nu_{\boldsymbol{f}} \veebar \bot)) \equiv \mathbf{F}\varphi \wedge \nu_{\boldsymbol{f}} \equiv \mathbf{F}\varphi$.

By 4.7.13(e) and the induction hypothesis,

$$\mathbf{T}(\varphi \wedge (\nu_{\boldsymbol{f}} \veebar \top)) \equiv \mathbf{T}\varphi \wedge \nu_{\boldsymbol{f}} \equiv \bigwedge(\nu_{\boldsymbol{u}} \mid \boldsymbol{u} \in A \cup B \cup C \cup \{\boldsymbol{f}\}).$$

By 4.7.13(c) and 4.7.8, and since $\boldsymbol{f} \notin B$,

$$\mathbf{D}(\varphi \wedge (\nu_{\boldsymbol{f}} \veebar \top)) \equiv \mathbf{D}\varphi \wedge \nu_{\boldsymbol{f}} \equiv \mathbf{D}\varphi.$$

By 4.7.12, $\mathbf{M}(\nu_{\boldsymbol{f}} \veebar \top) \equiv \mathbf{F}\nu_{\boldsymbol{f}}$, so by 4.7.13(d),

$$\mathbf{M}(\varphi \wedge (\nu_{\boldsymbol{f}} \veebar \top)) \equiv \mathbf{M}\varphi \veebar \kappa_{\boldsymbol{f}}.$$

Distribute $\kappa_{\boldsymbol{f}}$ into a normal form of $\mathbf{M}\varphi$. ¶

5.4 Lemma. Let $\varphi =$

$$\bigwedge_{a\in A} \nu_a \wedge \bigwedge_{b\in B} (\nu_b \veebar \bot) \wedge \bigwedge_{c\in C} (\nu_c \veebar \top)$$

and $\psi =$

$$\bigwedge_{d\in D} \nu_d \wedge \bigwedge_{e\in E} (\nu_e \veebar \bot) \wedge \bigwedge_{f\in F} (\nu_f \veebar \top)$$

be X-normal forms, with $A, B, C, D, E, F \subseteq L^n$. Let $R = (A\cup D)-(C\cup F)$, $S = (B\cup E)-(A\cup C\cup D\cup F)$, $T = C\cup F$, $U = A\cap D$, $V = B\cup(A\cap E)$, $W = C\cup(A\cap F)$. Then

(a) $\quad \varphi\wedge\psi \equiv \bigwedge_{r\in R} \nu_r \wedge \bigwedge_{s\in S} (\nu_s \veebar \bot) \wedge \bigwedge_{t\in T} (\nu_c \veebar \top)$,

(b) $\quad \varphi\veebar\psi \equiv \bigwedge_{u\in U} \nu_u \wedge \bigwedge_{v\in V} (\nu_v \veebar \bot) \wedge \bigwedge_{w\in W} (\nu_w \veebar \top)$.

Proof. By induction on the length of ψ.

(a) The case $\psi = 1$ ($D = E = F = \varnothing$) is trivial. For the case that one of D, E, F is a singleton while the other two are void, we note that since $\mathbf{M}\bot \equiv 0$, by (C3, 4) and 4.7.4, $\nu_u \wedge (\nu_u \veebar \bot) \equiv \nu_u \veebar (\nu_u \wedge \bot) \equiv \nu_u$, while

$$\nu_u \wedge (\nu_u \veebar \top) \equiv (\nu_u \veebar 0) \wedge (\nu_u \veebar \top) \equiv \nu_u \veebar (0\wedge\top) \equiv \nu_u \veebar \top$$

and similarly $(\nu_u \veebar \bot) \wedge (\nu_u \veebar \top) \equiv \nu_u \veebar \top$. For the rest the induction step is trivial.

(b) To begin with, let us suppose $\psi = \nu_d$. By 5.3 and 5.1,

$$(1) \qquad \mathbf{F} \bigwedge_{c\in C} (\nu_c \veebar \top) \equiv \mathbf{D} \bigwedge_{c\in C} (\nu_c \veebar \top) \equiv \bigwedge_{u\in L^n} \nu_u \equiv 0,$$

so by (C2)

$$\varphi\veebar\psi \equiv \Big(\big(\bigwedge_{a\in A} \nu_a \wedge \bigwedge_{b\in B} (\nu_b \veebar \bot)\big) \veebar \psi\Big) \wedge \Big(\bigwedge_{c\in C} (\nu_c \veebar \top) \veebar \psi\Big).$$

Repeating the argument we find

$$\bigwedge_{c\in C} (\nu_c \veebar \top) \veebar \psi \equiv \bigwedge_{c\in C} (\nu_c \veebar \top \veebar \psi) \equiv \bigwedge_{c\in C} (\nu_c \veebar \top)$$

by 4.7.2. For the other conjunct, first note that

$$(2) \qquad \mathbf{D} \bigwedge_{a\in A} \nu_a \equiv 0$$

by 5.3 and 5.1, and

$$(3) \qquad \mathbf{F} \bigwedge_{a\in A} \nu_a \wedge \mathbf{D} \bigwedge_{b\in B} (\nu_b \veebar \bot) \equiv \bigwedge_{u\in L^n} \nu_u \equiv 0.$$

So by (C2),

$$\big(\bigwedge_{a\in A} \nu_a \wedge \bigwedge_{b\in B} (\nu_b \veebar \bot)\big) \veebar \psi \equiv \big(\bigwedge_{a\in A} \nu_a \veebar \psi\big) \wedge \big(\bigwedge_{b\in B} (\nu_b \veebar \bot) \veebar \psi\big).$$

Recall that if $g \neq h$ then $\nu_g \veebar \nu_h \equiv 1$. So by 4.7.5 the first conjunct reduces to ν_d if $d \in A$, and to 1 otherwise. But since for any $G \subseteq L^n$, $\mathbf{F}\bigwedge(\nu_g \veebar \bot \mid g \in G) \equiv 0$, (C2) applies to the second conjunct, and ψ vanishes by 4.7.2.

Since $g \neq h$ implies $\varphi \veebar 1 \equiv \varphi \veebar \nu_g \veebar \nu_h$,

$$\varphi \veebar 1 \equiv \bigwedge_{b \in B}(\nu_b \veebar \bot) \wedge \bigwedge_{c \in C}(\nu_c \veebar \top).$$

Now consider $\varphi \veebar \nu_e \veebar \bot$. If $e \in A$,

$$\varphi \veebar \nu_e \veebar \bot \equiv \Big(\nu_e \wedge \bigwedge_{b \in B}(\nu_b \veebar \bot) \wedge \bigwedge_{c \in C}(\nu_c \veebar \top)\Big) \veebar \bot.$$

Since $\mathbf{T}\bot \equiv 0$ (4.4.20), by (C2) and 4.7.2 we get

$$(\nu_e \veebar \bot) \wedge \bigwedge_{b \in B}(\nu_b \veebar \bot) \wedge \bigwedge_{c \in C}(\nu_c \veebar \top).$$

Similarly if $e \notin A$,

$$\varphi \veebar \nu_e \veebar \bot \equiv \bigwedge_{b \in B}(\nu_b \veebar \bot) \wedge \bigwedge_{c \in C}(\nu_c \veebar \top)\Big) .$$

An analogous argument works for $\nu_e \veebar \top$.

In the remaining case ψ is a conjunction of nonvoid conjunctions ψ_1 and ψ_2. Then $\varphi \veebar \psi \equiv (\varphi \veebar \psi_1) \wedge (\varphi \veebar \psi_2)$, and we are done by the induction hypothesis and a bit of set algebra. ¶

6. Completeness

Let us call X-normal forms *distinct* if they do not have the same conjuncts.

Lemma 6.1. If φ and ψ are distinct X-normal forms, then $\mathbf{M} \nvDash \varphi \equiv \psi$.

Proof. Let φ and ψ be distinct X-normal forms. Then there are a dual criterion ν_a and distinct constants $\mathbf{c}$ and $\mathbf{d}$ such that for $\varphi' = \varphi \wedge (\nu_a \veebar \mathbf{c})$ and $\psi' = \psi \wedge (\nu_a \veebar \mathbf{d})$, $\mathcal{L} \vdash \varphi \equiv \varphi'$ and $\mathcal{L} \vdash \psi \equiv \psi'$. Now if $b \in L^n - \{a\}$, $\nu_b(a) = 1$; and $\nu_a(a) = 0$; so $\varphi'(a) = c$ and $\psi'(a) = d$, hence $\mathbf{M} \nvDash \varphi \equiv \psi$. ¶

Theorem 6.2. $\mathcal{L}$ is a complete axiomatization of the equational theory of $\mathbf{L}$.

Remark. The axioms of $\mathcal{L}$ are valid in $\mathbf{L}$, by propositions 4.1.1, 4.2.1, 4.3, 4.4.1, 4.5.1, 4.6 and 4.7.1.

Proof. If $\mathcal{L} \nvdash \varphi \equiv \psi$, then φ and ψ have distinct normal forms φ^n and ψ^n. Then $\mathbf{L} \nvDash \varphi^n \equiv \psi^n$; hence $\mathbf{L} \nvDash \varphi \equiv \psi$. ¶

7. Improvements and alternatives

There are twenty-five equations in the system $\mathcal{L}$, and many are ugly. One should like to have fewer axioms, and these expressing profound insights into the primitives $\wedge$, $\veebar$, $\downarrow$, $\top$ and $\bot$. *Just* fewer axioms is not so hard. Without any trouble we can bring down the number to five:

(i) $0 \veebar \mathbf{x} \equiv \mathbf{x}$

(ii) $1 \veebar \mathbf{x} \equiv 1$

(iii) $1 \wedge \mathbf{x} \equiv \mathbf{x}$

(iv) $\mathbf{x} \wedge \mathbf{y} \equiv \mathbf{y} \wedge \mathbf{x}$,

and a rather more extensive axiom (v) that needs some introduction. Number the axioms of $\mathcal{L}$ from 1 to 25, in the order in which they are introduced above. Let the i-th axiom be $\varphi_i \equiv \psi_i$, and define repeated conjunction $\bigwedge$ from $\wedge$. For each $i \in I := \{1,2,4,6,7,9,\ldots,25\}$, take a fresh variable $\mathbf{x}_i$. Then (v) is

$$\bigwedge_{i\in I}(\mathbf{x}_i \veebar \varphi_i) \equiv \bigwedge_{i\in I}(\mathbf{x}_i \veebar \psi_i).$$

From (v) we get $\varphi_j \equiv \psi_j$ by substituting 0 for $\mathbf{x}_j$ and 1 for all $\mathbf{x}_i$ with $i \neq j$, and simplifying by means of (i-iv). So $\mathcal{L}$ follows from these five equations. Conversely, (i-v) follow from $\mathcal{L}$ since they hold in $\mathbf{L}$ and $\mathcal{L}$ is complete.

There may also be better primitives, in some sense or for some purpose, than the set employed above. (It is well known that there exists a single functionally complete connective, which by a remark in the introduction is finitely axiomatizable.)

8. Related work

The four truth values of L may perhaps be said to have made their first appearance in the work of Goddard and Routley [GR]; their ordering with respect to information content is due to Belnap [B]. Fitting [F] has studied four-valued extensions of Kleene's three-valued logics. There has been extensive study of this kind of four-valued logic in relation to the semantics of natural language, for which I refer to [M] and [LL]. Moreover, Muskens has formalized the logic corresponding to his theory of semantics (as yet unpublished).

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