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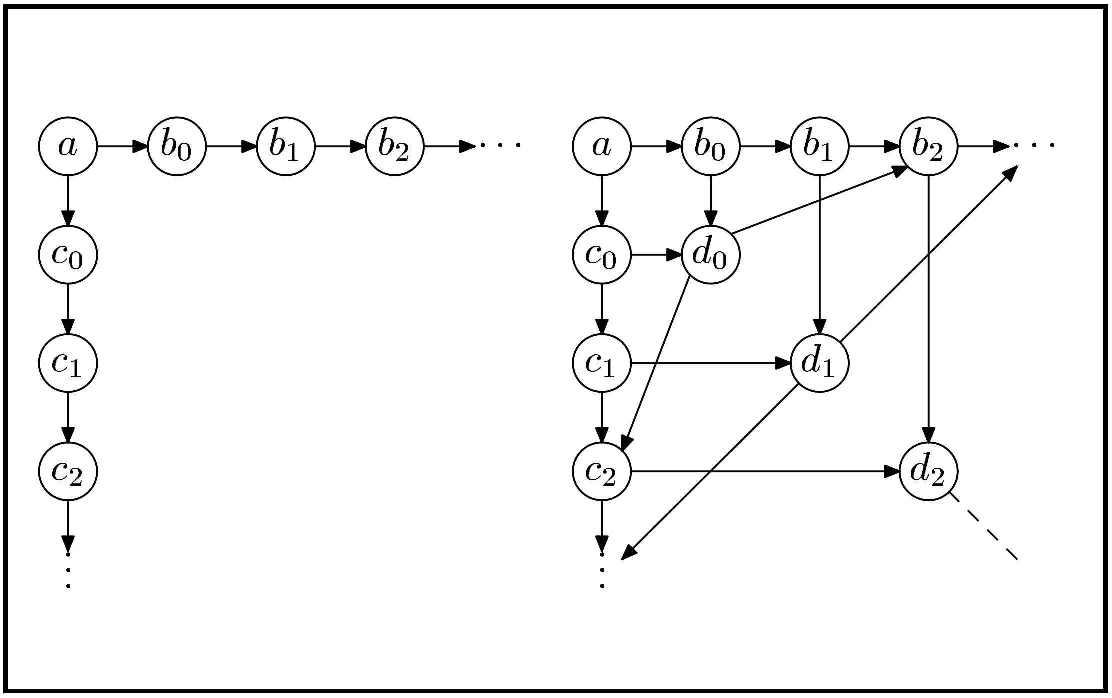

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Transformations of Reduction Systems

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Abstract

We consider transformations of reduction systems in an abstract setting. We study some sets of correctness criteria for such transformations, adapt a notion of simulation proposed by Kamperman and Walters, and show that the resulting ω-simulation behaves well w.r.t. the criteria. We apply our results in an investigation of a transformation proposed by Thatte, and prove that this transformation preserves semi-completeness for weakly persistent systems.

1 Introduction

With the emergence of the use of Reduction Systems as semantical basis of programming paradigms, the use of transformation methods for Reduction Systems is growing. Some systems are more suitable for execution on a machine than others, for instance because they have a normalizing reduction strategy that is easily decidable, or because they are in a specific format. Transformation methods are then used to translate an arbitrary system into such a suitable system.

Evidently, such a transformation should not be applied if it does not induce some sensible connection between the transform and its original. Only if a transformation fulfills a number of predefined criteria, it should be called correct. Naturally, the answer to the question of whether the connection between transform and original is a sensible one is rather subjective. It depends largely on the application and little can be said in general. However, if the transformation is part of an implementation of Reduction Systems on automata, then there are some widely accepted intuitions:

- In Concurrency Theory and Automata Theory much research has been done towards satisfactory connections, there called "simulation relations" (see for instance [LV95]). These relations induce a preorder on the class of automata, such that if automaton A simulates automaton B, then A can be seen as an implementation of B.
- In papers dealing with transformations for Reduction Systems and their correctness, it is usually proved that the normal form relations of the original and its transform are related [Tha85, Tha88, Ver95a, Ver95b]. This is because the normal form relation of a Reduction System is its computational content; it relates input and result.

Kamperman and Walters initiated a new approach towards correctness proofs of transformations for Reduction Systems [KW96, Kam96]. In order to make correctness of their transformation for Term Rewriting Systems plausible, they prove that the transform "simulates" the original. Fokkink and Van de Pol prove that the simulation relation of Kamperman and Walters implies preservation of the normal form relation. They define a notion of *correct transformation* in which a connection between the normal form relations of original and transform figures as a principal component [FP96].

In this paper we set up a general framework in which correctness criteria can be expressed and which fits the various notions in the literature. As a derivative of the notion of Kamperman and Walters, we define a notion of simulation that we call ω-simulation. We study its preservation properties and prove that the existence of a termination preserving ω-simulation implies correctness as defined by Fokkink and Van de Pol.

We apply our results to a transformation proposed by Thatte. This transformation was introduced in 1985 in [Tha85] for regular Term Rewriting Systems. In [Tha88], Thatte claimed that it preserved confluence and (up to some translation on terms) the normal form relation for nonoverlapping systems, but it was shown by Verma in [Ver95b] that this claim was false.

Verma, for his part, tried to fix Thatte's proof by imposing an extra condition (weak persistence) on the domain of the transformation [Ver95b]. However, we discovered some fundamental errors in his proof (see appendix A) and after we had notified Verma of these, he found a counterexample [Ver96].

We investigate Thatte's transformation for a number of classes of Term Rewriting Systems. We give an improved definition of weak persistence and show that Thatte's transform preserves confluence for the particular classes of weakly persistent systems that Verma mentions in [Ver95b]. Finally, we will obtain a proof that Thatte's transformation preserves semi-completeness.

Acknowledgements. We thank Wan Fokkink, Jasper Kamperman and Jaco van de Pol for the many discussions on correctness criteria. Furthermore, we thank Jan Bergstra for stimulating us to continue this work.

2 Preliminaries

An *Abstract Reduction System* (ARS) is a pair $\langle A, (\to_\alpha)_{\alpha \in I}\rangle$ consisting of a set A and a family of binary relations $\to_\alpha$ on A (indexed by some set I), also called *reduction* or *rewrite relations*. We reserve $a, b, c, \ldots$ (possibly subscripted) for elements of A, and we use $\to_\alpha, \to_\beta, \to_\gamma, \ldots$ or simply $\to$ to refer to rewrite relations. $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ (sometimes primed or subscripted) will be used to denote ARSs.

If for $a, b \in A$ we have $\langle a, b\rangle \in \to_\alpha$ then we write $a \to_\alpha b$ and call b a 'one-step (α-)reduct of a'.

The transitive closure of $\to_\alpha$ is written as $\to_\alpha^+$. So $a \to_\alpha^+ b$ if there is a finite sequence of 'reduction steps' (a reduction) $a = a_0 \to_\alpha a_1 \to_\alpha \cdots \to_\alpha a_n = b$, where $n \geq 1$. We also write $a \to_\alpha^n b$ if we want to be explicit about the number of reduction steps.

We write $a \to_\alpha^0 b$ if $a = b$ and denote by $\twoheadrightarrow_\alpha$ or $\to_\alpha^*$ the transitive reflexive closure of $\to_\alpha$. That is, $a \twoheadrightarrow_\alpha b$ iff $a \to_\alpha^n b$ for some $n \in \omega$. If $a \twoheadrightarrow_\alpha b$, then b is called a *reduct* of a, and a is said to *reduce to* b.

We write $\leftarrow_\alpha$ for the converse relation of $\to_\alpha$ and this same convention ports to all other 'reduction relations' that we will define. So, for instance, $\twoheadleftarrow_\alpha$ and $\leftarrow_\alpha^+$ denote the converses of $\twoheadrightarrow_\alpha$ and $\to_\alpha^+$.

If an ARS is of the form $\langle A, \to\rangle$, i.e. it involves only one reduction relation ($|I| = 1$), then we call it *simple*. Each ARS $\mathfrak{A} = \langle A, (\to_\alpha)_{\alpha \in I}\rangle$ is straightforwardly mapped onto a simple ARS $\langle A, \bigcup_{\alpha \in I} \to_\alpha\rangle$; the latter is called the *simple* ARS *underlying* $\mathfrak{A}$.

If X is a set and $\mathcal{R}$ a binary relation we denote the set $\{y \mid \exists x \in X(x\mathcal{R}y)\}$ by $[X]_\mathcal{R}$. In the particular case that $\mathcal{R}$ is the transitive reflexive closure $\twoheadrightarrow$ of the reduction relation of an ARS $\mathfrak{A} = \langle A, \to \rangle$ and $X \subseteq A$, we denote $[X]_{\mathcal{R}}$ by $[X]_{\mathfrak{A}}$, or $[X]$ when $\mathfrak{A}$ is understood. The substructure of $\mathfrak{A}$ with domain $[X]$ we denote by $\mathfrak{A}[X]$; instead of $\mathfrak{A}[\mathsf{range}(\phi)]$ we write $\mathfrak{A}[\phi]$. If $y \in [X]_{\mathcal{R}}$, we say that y is *reachable* (from X by $\mathcal{R}$).

Let $\mathfrak{A} = \langle A, \to \rangle$ be an ARS and $a \in A$.

If $\forall b, c \in A\ \exists d \in A\ (c \leftarrow a \to b \Longrightarrow c \twoheadrightarrow d \twoheadleftarrow b)$, then a is *weakly confluent* (or: *weakly Church-Rosser*) (notation: $\mathfrak{A} \models \mathsf{WCR}(a)$); if $\forall a \in A (\mathfrak{A} \models \mathsf{WCR}(a))$,then $\mathfrak{A}$ is *weakly confluent* ($\mathfrak{A} \models \mathsf{WCR}$).

If $\forall b, c \in A\ \exists d \in A\ (c \twoheadleftarrow a \twoheadrightarrow b \Longrightarrow c \twoheadrightarrow d \twoheadleftarrow b)$, then a is *confluent* (or: *Church-Rosser*) notation: $\mathfrak{A} \models \mathsf{CR}(a)$); if $\forall a \in A (\mathfrak{A} \models \mathsf{CR}(a))$, then $\mathfrak{A}$ is *confluent* ($\mathfrak{A} \models \mathsf{CR}$).

We say that a is a *normal form* if there is no $b \in A$ such that $a \to b$. Furthermore, *a has a normal form* if $a \twoheadrightarrow n$ for some normal form $n \in A$; in this case a is *weakly terminating* (or: *weakly normalizing*) (notation: $\mathfrak{A} \models \mathsf{WN}(a)$). If $\forall a \in A (\mathfrak{A} \models \mathsf{WN}(a))$, then $\mathfrak{A}$ is *weakly normalizing* ($\mathfrak{A} \models \mathsf{WN}$).

If there is no sequence $(a_n)_{n \in \omega}$ such that $a_0 = a$ and $a_n \to a_{n+1}$, then a is *terminating* (or: *strongly normalizing, noetherian*) (notation: $\mathfrak{A} \models \mathsf{SN}(a)$). If $\forall a \in A (\mathfrak{A} \models \mathsf{SN}(a))$, then $\mathfrak{A}$ is *strongly normalizing* ($\mathfrak{A} \models \mathsf{SN}$).

If $\forall b, c \in A (a \twoheadrightarrow b \mathbin{\&} a \twoheadrightarrow c \mathbin{\&} b, c$ are normal forms $\Longrightarrow b = c)$, then a has the *unique normal form property* (notation: $\mathfrak{A} \models \mathsf{UN}(a)$). If $\forall a \in A (\mathfrak{A} \models \mathsf{UN}(a))$, then $\mathfrak{A}$ has the *unique normal form property* ($\mathfrak{A} \models \mathsf{UN}$, $\mathfrak{A}$ is *uniquely normalizing*).

We also write $\mathfrak{A} \not\models \mathsf{WCR}(a)$ if $a \in A$ is not weakly confluent, etc. All these properties generalize to arbitrary ARSs as follows:

> $\langle A, (\to_\alpha)_{\alpha \in I} \rangle$ has property X if its underlying simple ARS $\langle A, \bigcup_{\alpha \in I} \to_\alpha \rangle$ has property X.

A confluent and terminating ARS is called *complete*; a confluent and weakly terminating ARS is called *semi-complete*.

Trivially, the implications $\mathsf{CR} \Longrightarrow \mathsf{UN}$ and $\mathsf{UN} \mathbin{\&} \mathsf{WN} \Longrightarrow \mathsf{CR}$ hold. Another well-known implication is $\mathsf{SN} \mathbin{\&} \mathsf{WCR} \Longrightarrow \mathsf{CR}$ (Newman's Lemma).

The set of normal forms of $\mathfrak{A}$ we denote by $\mathsf{NF}(\mathfrak{A})$. Furthermore, $\mathsf{nf}_{\mathfrak{A}}$ denotes the *normal form relation of* $\mathfrak{A}$, i.e. $\mathsf{nf}_{\mathfrak{A}} = \{\langle a, b \rangle \in A \times \mathsf{NF}(\mathfrak{A}) \mid b$ is a normal form of $a\}$. If $a \in A$ reduces to a unique normal form n, then we denote this n by $\mathsf{nf}_{\mathfrak{A}}(a)$; if $\mathfrak{A} \models \mathsf{UN}$ then $\mathsf{nf}_{\mathfrak{A}}$ is a (partial) function. If $\mathfrak{A}$ is clear from the context, then we just write nf instead of $\mathsf{nf}_{\mathfrak{A}}$. We use NF_α, nf_α for normal forms w.r.t. a particular relation $\to_\alpha$ of an ARS $\langle A, (\to_\alpha)_{\alpha \in I} \rangle$.

3 Correctness Criteria

A criterion for correctness of some transformation of ARS $\mathfrak{A} = \langle A, \to_\alpha \rangle$ into $\mathfrak{B} = \langle B, \to_\beta \rangle$ essentially consists of two parts.

On the one hand, the elements of A must be in some way connected to those of B. In order to establish such a connection we will use a pair of functions, one from A to B and one from B to A, and we will identify the transformation with this pair of functions. All possibilities we encountered in literature fit into this framework.

On the other hand, there must be some connection between the reduction relations $\to_\alpha$ and $\to_\beta$. Especially this part is open to discussion. Until now there has not been much consensus about how to express this connection and we do not believe that such consensus is to be expected in the near future. We classify the various opinions we found roughly into two categories: the *direct approach* and the *indirect approach*.

By the direct approach we mean that the main interest lies in proving that certain properties are invariant under the transformation, such as the normal form relation, confluence, and termination. By its abstract formulation this approach hardly gives any direction for the prover, but it seems to be more powerful in the sense that more transformations can be proven "correct". However, much information about the exact relation that exists between the reduction relations of the original and its transform remains rather implicit.

The indirect approach has more to do with the structure of the reduction relations in terms of the reduction steps. Some connection between the reduction steps is established. Therefore, we could also call this approach *operational* or *behavioural*. It leads to notions that resemble the notions of simulation that occur in Automata Theory and Concurrency Theory (see for example [LV95]).

We will describe both approaches independently in the following sections. However, we feel that the division does not have to be all that strict. To emphasize this, we will apply a mix of both approaches in our example transformation. The preservation of reduction graphs in combination with a specific mapping between the elements of the two systems has a number of pleasant consequences in terms of the preservation of properties. We will describe and prove these in section 4.

As we did with the properties CR, SN etc., we will define all our correctness notions for simple ARSs. They generalize to arbitrary ARSs with more than one reduction relation in the same way as in section 2.

3.1 Simulations

To express a connection between the objects of the original system $\langle A, \to_\alpha \rangle$ and its transform $\langle B, \to_\beta \rangle$ we use a pair of mappings $\langle \phi, \psi \rangle$. These mappings provide the there and back translations from the original to its transform (ϕ) and from the transform to its original (ψ).[1] We will demand that these translations are consistent with each other for all $a \in A$, in the sense that their composition $\psi \circ \phi$ is the identity function on A.

Unless otherwise stated we will adapt the convention that $\mathfrak{A}$ refers to the ARS $\langle A, \to_\alpha \rangle$, $\mathfrak{B}$ to $\langle B, \to_\beta \rangle$, etc.

Definition 3.1 (inspired by **[FP96]**) Let $\mathfrak{A}$ and $\mathfrak{B}$ be ARSs.

A *simulation* of $\mathfrak{A}$ by $\mathfrak{B}$ is a pair $\langle \phi, \psi \rangle$ of a function $\phi : A \to B$ and a partial function $\psi : B \to A$ such that

1. $\mathsf{range}(\phi) \subseteq \mathsf{domain}(\psi)$; and
2. $\psi \circ \phi = \mathsf{id}(A)$. $\dashv$

We call $\mathfrak{A}$ the *domain* of the simulation $\langle \phi, \psi \rangle$ and $\mathfrak{B}$ the *codomain* or the transform. We say $\mathfrak{B}$ is reachable if $\mathfrak{B} = \mathfrak{B}[\phi]$.

Often in the sequel we shall have $\psi = \phi^{-1}$; we shall write ϕ to refer to the simulation $\langle \phi, \phi^{-1} \rangle$. In another important special case we have $A \subseteq B$ and $\phi = \mathsf{id}(A)$ (see [Kam96]).

3.2 Preservation of Properties

Most authors of papers on transformations require some form of preservation of the normal form relation. The following definition has been adapted from Fokkink and Van de Pol [FP96].

Definition 3.2 (Preservation of Normal Forms) Let $\langle \phi, \psi \rangle$ be a simulation of $\mathfrak{A}$ by $\mathfrak{B}$ such that $\mathsf{NF}(\mathfrak{B}[\phi]) \subseteq \mathsf{domain}(\psi)$.

$\langle \phi, \psi \rangle$ *preserves the normal form relation* if for all $a \in A$:

$$\{\psi(n) \mid n \in \mathsf{nf}_{\mathfrak{B}}[\phi(a)]\} = \mathsf{nf}_{\mathfrak{A}}[a]$$

[1] In [FP96] these mappings are also called `parse` and `print`, respectively.

$\langle \phi, \psi \rangle$ *weakly preserves the normal form relation* if for all $a \in A$:

$$\{\psi(n) \mid n \in \mathsf{nf}_{\mathfrak{B}}[\phi(a)]\} \subseteq \mathsf{nf}_{\mathfrak{A}}[a] \qquad \dashv$$

It is easy to see that these requirements do not guarantee that $\mathfrak{B}$ is strongly normalizing or confluent if $\mathfrak{A}$ is, even if $\mathfrak{B} = \mathfrak{B}[\phi]$. Accordingly, Fokkink and Van de Pol consider a further requirement on termination behaviour in $\mathsf{range}(\phi)$, while Thatte [Tha85, Tha88] and Verma [Ver95b] base their correctness arguments on the preservation of confluence.

Definition 3.3 (Preservation of Properties) Let $\langle \phi, \psi \rangle$ be a simulation of $\mathfrak{A}$ by $\mathfrak{B}$ and suppose X is a property defined for elements of ARSs (i.p. $\mathsf{X} \in \{\mathsf{SN}, \mathsf{WN}, \mathsf{CR}, \mathsf{WCR}, \mathsf{UN}\}$).
$\langle \phi, \psi \rangle$ *preserves* X if for all $a \in A$: $\mathfrak{A} \models \mathsf{X}(a) \Longrightarrow \mathfrak{B} \models \mathsf{X}(\phi(a))$. $\dashv$

The reachable transform of a weakly normalizing ARS by a weak termination preserving simulation need not be weakly normalizing. However, for termination preserving transformations we have:

Proposition 3.4 Let $\mathfrak{B}$ be the reachable transform of $\mathfrak{A}$ by $\langle \phi, \psi \rangle$.
If $\mathfrak{A} \models \mathsf{SN}$ and $\langle \phi, \psi \rangle$ preserves termination, then $\mathfrak{B} \models \mathsf{SN}$. $\square$

Proposition 3.5 Let $\mathfrak{B}$ be the reachable transform of $\mathfrak{A}$ by $\langle \phi, \psi \rangle$.
If $\mathfrak{A} \models \mathsf{CR}$ and $\langle \phi, \psi \rangle$ preserves confluence, then $\mathfrak{B} \models \mathsf{CR}$.

Proof By contraposition. Suppose $\mathfrak{B} \not\models \mathsf{CR}$. Then $\mathfrak{B} \not\models \mathsf{CR}(b)$ for some $b \in B$, and since $\mathfrak{B} = \mathfrak{B}[\phi]$, there is some $a \in A$ such that $\phi(a) \twoheadrightarrow_\beta b$. Clearly $\mathfrak{B} \not\models \mathsf{CR}(\phi(a))$, hence $\mathfrak{A} \not\models \mathsf{CR}(a)$. $\square$

The analogous result for weak confluence does not hold.

Example 3.1 Consider the ARSs $\mathfrak{A}_1 = \langle \{a, b\}, \{a \to_\alpha b\} \rangle$ and $\mathfrak{A}_2 = \langle \{a, b, c, d\}, \{a \to_\beta c,\ c \to_\beta b,\ c \to_\beta d\} \rangle$ with the simulation $\mathsf{id}(\{a, b\})$. Note that $\mathfrak{A}_1 \models \mathsf{WCR}$ and $\mathfrak{A}_2 \models \mathsf{WCR}(a)$, $\mathfrak{A}_2 \models \mathsf{WCR}(b)$, so $\mathsf{id}(\{a, b\})$ preserves weak confluence. Also $\mathfrak{A}_2$ is reachable; but $\mathfrak{A}_2 \not\models \mathsf{WCR}(c)$.

Fokkink and Van de Pol call $\mathfrak{B}$ a correct transformation of $\mathfrak{A}$ if there isa simulation $\langle \phi, \psi \rangle$ that preserves the normal form relation and termination.

3.3 Preservation of Reduction Graphs

Kamperman and Walters advocate a form of preservation of reduction graphs as a correctness criterion. They call a simulation $\langle \phi, \psi \rangle$ of $\mathfrak{A}$ by $\mathfrak{B}$ *sound* if each reduction in $\mathfrak{B}$ simulates a reduction in $\mathfrak{A}$ and *complete* if each reduction in $\mathfrak{A}$ is actually simulated by a reduction in $\mathfrak{B}$. For their original definitions we refer to [Kam96]. Their approach is interesting, because it turns out that if one wants correctness via the direct approach of the previous section one often needs to prove soundness and completeness as lemmas (see [Tha85, Ver95b]).

We will define a notion of *preservation of reduction graphs* that is based on the notions of Kamperman and Walters, but differs slightly. In the next section we will provide further justification for the claim that such reduction conservation in combination with a simulation of the form $\langle \phi, \phi^{-1} \rangle$ is useful as a correctness criterion, by studying its consequences.

Definition 3.6 (Preservation of Reduction Graphs) A simulation $\langle \phi, \psi \rangle$ of $\mathfrak{A}$ by $\mathfrak{B}$ *preserves reduction graphs* if it satisfies

1. if $a \to_\alpha b$ $(a, b \in A)$, then $\phi(a) \to_\beta^+ \phi(b)$;

2. if $a \to_\beta^+ b$ for $a \in \mathsf{domain}(\psi)$ and $b \in B$, then $\exists c \in \mathsf{domain}(\psi)(\psi(a) \to_\alpha^+ \psi(c) \wedge b \twoheadrightarrow_\beta c)$.

$\dashv$

The first condition above is a generalization of completeness as in [Kam96]. One obtains the soundness condition of [Kam96] by substituting $\twoheadrightarrow_\beta$, $\twoheadrightarrow_\alpha$ for $\to_\beta^+$, $\to_\alpha^+$ in the second condition above.

4 ω-Simulation

We study the implications of preservation of reduction graphs for the preservaton of properties as in Definition 3.3. It turns out that for such a simulation mapping we get the preservation of the normal form relation and confluence for free. First we introduce some extra terminology.

Definition 4.1 (ω-simulation[2]) An ω-simulation is a reduction graph preserving simulation of the form $\langle \phi, \phi^{-1} \rangle$.

If ϕ is an ω-simulation of $\mathfrak{A}$ by $\mathfrak{B}$, then we write $\mathfrak{A} \Rightarrow_\omega^{\phi} \mathfrak{B}$; we say $\mathfrak{B}$ ω-simulates $\mathfrak{A}$ ($\mathfrak{A} \Rightarrow_\omega \mathfrak{B}$) if there is an ω-simulation of $\mathfrak{A}$ and $\mathfrak{B}$. $\dashv$

4.1 A Preorder

If the composition of two correct transformations is again a correct transformation, i.e. '... is correctly simulated by ...' is a transitive relation of ARSs, it may be considerably easier to prove complicated transformations correct. We could view a transformation as a composition of transformations that are easier to deal with. This method is known as 'stepwise refinement' (see [LV95]).

All criteria defined in section 3.2 trivially induce preorders (transitive reflexive relations) on the class of ARSs. It is moreover easy to see that ω-simulation is reflexive. Transitivity is a little less clear.

Lemma 4.2 If $\mathfrak{A} \Rightarrow_\omega^{\phi} \mathfrak{B} \Rightarrow_\omega^{\psi} \mathfrak{C}$ then $\mathfrak{A} \Rightarrow_\omega^{\psi \circ \phi} \mathfrak{C}$.

Proof Let $\chi = \psi \circ \phi$.

It is clear χ satisfies the first conditions of Definition 3.6. We verify the second condition. Assume $a \to_\gamma^+ b$ for $a \in \mathsf{range}(\chi)$ and $b \in C$. We need to derive the existence of a $c \in \mathsf{range}(\chi)$ such that $\chi^{-1}(a) \to_\alpha^+ \chi^{-1}(c)$ and $b \twoheadrightarrow_\gamma c$. Because $\mathfrak{B} \Rightarrow_\omega^{\psi} \mathfrak{C}$, there is a $c_2 \in \mathsf{range}(\psi)$ with $\psi^{-1}(a) \to_\beta^+ \psi^{-1}(c_2)$ and $b \twoheadrightarrow_\gamma c_2$, and because $\mathfrak{A} \Rightarrow_\omega^{\phi} \mathfrak{B}$, there is a $c_1 \in \mathsf{range}(\phi)$, such that $\psi^{-1}(c_2) \twoheadrightarrow_\beta c_1$ and $\phi^{-1}(\psi^{-1}(a)) \to_\alpha^+ \phi^{-1}(c_1)$. It now remains to show that $b \twoheadrightarrow_\gamma \psi(c_1)$. Note that from $\psi^{-1}(c_2) \twoheadrightarrow_\beta c_1$ and the first condition satisfied by $\mathfrak{B} \Rightarrow_\omega^{\psi} \mathfrak{C}$ we find that $c_2 \twoheadrightarrow_\gamma \psi(c_1)$. Combining this reduction with $b \twoheadrightarrow_\gamma c_2$, we find $b \twoheadrightarrow_\gamma \psi(c_1)$, so we may take $c = \psi(c_1)$. $\square$

Corollary 4.3 ω-simulation induces a preorder on the class of ARSs. $\square$

4.2 Preservation of Normal Forms

The existence of an ω-simulation implies a close connection between the normal form relations of the two systems.

Proposition 4.4 If $\mathfrak{A} \Rightarrow_\omega^{\phi} \mathfrak{B}$, then for $a, b \in A$ we have:

i. $n \in \mathsf{NF}(\mathfrak{A})$ iff $\phi(n) \in \mathsf{NF}(\mathfrak{B})$;

[2] We use the name "ω-simulation" because of the familiarity of this notion with the simulation notions that occur in Concurrency Theory (see [LV95]). The ω emphasizes the fact that possibly one step in the original is simulated by infinitely many steps in the transform.

ii. if $b \twoheadrightarrow_\beta n \in \mathsf{NF}(\mathfrak{B})$, and $b \in \mathsf{range}(\phi)$, then $n \in \mathsf{range}(\phi)$ and $\phi^{-1}(b) \twoheadrightarrow_\alpha \phi^{-1}(n)$.

PROOF

i. If $\phi(n) \notin \mathsf{NF}(\mathfrak{B})$, then $\phi(n) \to_\beta b$ for some $b \in B$, and using the second condition of Definition 3.6 we find that there must be a $c \in \mathsf{range}(\phi)$, such that $n \to_\alpha^+ \phi^{-1}(c)$, so $n \notin \mathsf{NF}(\mathfrak{A})$.

   On the other hand, if $n \notin \mathsf{NF}(\mathfrak{A})$ then there exists $a \in A$ such that $n \to_\alpha a$. By the first condition of Definition 3.6 we find $\phi(n) \to^+ \phi(a)$, so $\phi(n) \notin \mathsf{NF}(\mathfrak{B})$.

ii. Assume $b \twoheadrightarrow_\beta n \in \mathsf{NF}(\mathfrak{B})$ and $b \in \mathsf{range}(\phi)$.

   If $b = n$, then trivially $\phi^{-1}(b) \twoheadrightarrow_\alpha \phi^{-1}(n)$. If $b \to_\beta^+ n$, then by the second condition of Definition 3.6 $\exists c {\in} \mathsf{range}(\phi)(\phi^{-1}(b) \to_\alpha^+ \phi^{-1}(c) \wedge n \twoheadrightarrow_\beta c)$, and $n = c$ since $n \in \mathsf{NF}(\mathfrak{B})$. □

COROLLARY 4.5 If $\mathfrak{A} \lessapprox^\phi \mathfrak{B}$, then ϕ preserves the normal form relation.

4.3 Preservation of Confluence

An arbitrary simulation $\langle \phi, \psi \rangle$ that preserves reduction graphs does not necessarily preserve confluence:

EXAMPLE 4.1 Consider ARSs $\langle \{a, b\}, \{a \to_\alpha b\} \rangle$ and $\langle \{c, d_1, d_2\}, \{c \to_\beta d_1,\ c \to_\beta d_2\} \rangle$ and define ϕ and ψ by:

$$\begin{array}{rclcrcl} \phi(a) & = & c & \quad & \psi(c) & = & a \\ \phi(b) & = & d_1 & & \psi(d_1) & = & b \\ & & & & \psi(d_2) & = & b \end{array}$$

It is clear that $\langle \phi, \psi \rangle$ preserves reduction graphs as defined in Definition 3.6. Furthermore, it is clear that $\mathfrak{B}$ is reachable. However, $\mathfrak{A} \models \mathsf{CR}(a)$ while it is not the case that $\mathfrak{B} \models \mathsf{CR}(\phi(a))$.

The problem in the above counterexample lies in the fact that $\psi \neq \phi^{-1}$. If we assume a mapping of the form $\langle \phi, \phi^{-1} \rangle$ then we exclude the above example and w.r.t. confluence the situation takes a pleasant turn.

THEOREM 4.6 (CONFLUENCE)
If $\mathfrak{A} \lessapprox^\phi \mathfrak{B}$, then we have for all $a \in A$: $\mathfrak{A} \models \mathsf{CR}(a) \Longrightarrow \mathfrak{B} \models \mathsf{CR}(\phi(a))$.

PROOF Fix an $a \in A$ and assume $\mathsf{CR}(a)$ and $c \twoheadleftarrow_\beta \phi(a) \twoheadrightarrow_\beta b$, with $b, c \in [\mathsf{range}(\phi)]_{\mathfrak{B}}$. If $\phi(a) \to_\beta^0 b$, or $\phi(a) \to_\beta^0 c$ then there is nothing to prove, so assume $\phi(a) \to_\beta^+ b$ and $\phi(a) \to_\beta^+ c$. Then by the second condition of Definition 3.6 there are $d_1, d_2 \in A$ such that $a \to_\alpha^+ d_1$, $a \to_\alpha^+ d_2$, $b \twoheadrightarrow_\beta \phi(d_1)$ and $b \twoheadrightarrow_\beta \phi(d_2)$. Because $\mathfrak{A} \models \mathsf{CR}(a)$, d_1 and d_2 must have a common reduct d, i.e. $d_1 \twoheadrightarrow_\alpha d \twoheadleftarrow_\alpha d_2$, which, by the first condition of Definition 3.6 and a simple induction argument, implies that we also have $\phi(d_1) \twoheadrightarrow_\beta \phi(d) \twoheadleftarrow_\beta \phi(d_2)$. In other words, b and c have a common reduct in $\mathfrak{B}$, so $\mathfrak{B} \models \mathsf{CR}(\phi(a))$. □

The reverse does not hold:

EXAMPLE 4.2 Consider the following ARSs (see also Figure 1 below):

$$\begin{array}{rcll} \mathfrak{A} & = & \langle & \{a\} \cup \{b_i, c_i \mid i \in \omega\}, \\ & & & \{a \to_\alpha b_0,\ a \to_\alpha c_0\} \cup \{b_i \to_\alpha b_{i+1},\ c_i \to_\alpha c_{i+1} \mid i \in \omega\} \quad \rangle \text{ and} \\ \mathfrak{A}' & = & \langle & \{a\} \cup \{b_i, c_i, d_i \mid i \in \omega\}, \\ & & & \{a \to_\beta b_0,\ a \to_\beta c_0\} \cup \{b_i \to_\beta b_{i+1},\ c_i \to_\beta c_{i+1} \mid i \in \omega\} \cup \\ & & & \{b_i \to_\beta d_i, c_i \to_\beta d_i, d_i \to_\beta b_{i+2}, d_i \to_\beta c_{i+2} \mid i \in \omega\} \quad \rangle \end{array}$$

Clearly, all objects of $\mathfrak{A}'$ are reachable. We show that $\mathfrak{A} \precsim^{\mathsf{id}(A)} \mathfrak{A}'$.

Note that each reduction in $\mathfrak{A}$ is also a reduction in $\mathfrak{A}'$ (this verifies the first condition) and that each d_i has a reduction to all b_j, c_j with $j > i+1$. This means that:

1. each $a \to_\beta^+ d_i$ can be followed by $d_i \to_\beta b_{i+2}$ to simulate $a \to_\alpha^+ b_{i+2}$;
2. each $b_i \to_\beta^+ d_j$ can be followed by $d_j \to_\beta b_{j+2}$ to simulate $b_i \to_\alpha^+ b_{j+2}$;
3. each $c_i \to_\beta^+ d_j$ can be followed by $d_j \to_\beta c_{j+2}$ to simulate $c_i \to_\alpha^+ c_{j+2}$.

Moreover it is clear that $\mathsf{CR}(a)$ does not hold in $\mathfrak{A}$, while in $\mathfrak{A}'$ each pair of elements has a common reduct.

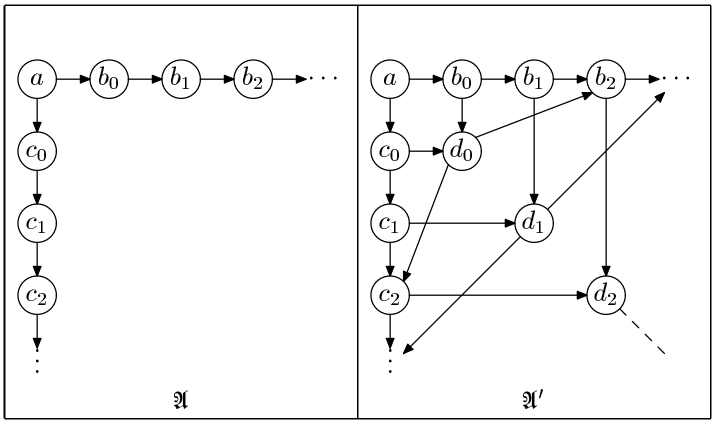

Figure 1: Counterexample: ω-simulation does not preserve confluence in the reverse direction.

Next, we show that ω-simulation does not preserve weak confluence in either direction. By Theorem 4.6 we already know that we need a system with at least one element that is WCR, but not CR. We use the famous counterexample by R. Hindley that shows that WCR does not imply CR.

Example 4.3 Consider the following ARSs (see also Figure 2):

$$\begin{aligned}
\mathfrak{A} &= \langle \{a,b,c,d\}, \{b \to_\alpha a,\ b \to_\alpha c,\ c \to_\alpha b,\ c \to_\alpha d\} \rangle \\
\mathfrak{A}' &= \langle \{a,b,c,d\}, \{b \to_\beta a,\ b \to_\beta c,\ c \to_\beta b,\ c \to_\beta d,\ b \to_\beta d\} \rangle
\end{aligned}$$

It is easy to see that $\mathfrak{A} \precsim^{\phi} \mathfrak{A}'$, as well as $\mathfrak{A}' \precsim^{\phi} \mathfrak{A}$, where $\phi = \mathsf{id}(\{a,b,c,d\})$. Note that operationally, $b \to_\beta d$ adds nothing to the original system. It is just a shortcut for the reduction $b \to_\alpha c \to_\alpha d$.

Neither of the systems is CR, but obviously $\mathfrak{A} \models \mathsf{WCR}$, while $\mathfrak{A}' \models \mathsf{WCR}(b)$ does not hold; we have $a \leftarrow_\beta b \to_\beta d$, but a and d have no common reduct.

4.4 Preservation of Termination

Unfortunately, the existence of an ω-simulation does not imply that the simulation map preserves termination. We obtain the following counterexample:

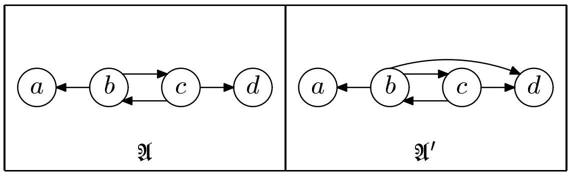

Figure 2: Counterexample: ω-simulation does not preserve WCR in either direction.

Example 4.4 Let $\mathfrak{A} = \langle \{a, c\}, \{a \to_\alpha c\}\rangle$ and $\mathfrak{A}' = \langle \{a, b, c\}, \{a \to_\beta b,\ b \to_\beta b,\ b \to_\beta c\}\rangle$ be ARSs. Then $\mathfrak{A} \lesssim^{\phi} \mathfrak{A}'$, with $\phi = \mathsf{id}(\{a, c\})$, because the only reduction possible in $\mathfrak{A}$ is $a \to_\alpha c$ and in $\mathfrak{A}'$ we have $a \to^+_\beta c$ by $a \to_\beta b \to_\beta c$ and that $b \twoheadrightarrow_\beta c$, so all reductions $a \to^+_\beta b$, i.e.

$$\begin{array}{l} a \to_\beta b \\ a \to_\beta b \to_\beta b \\ a \to_\beta b \to_\beta b \to_\beta b \\ \vdots \end{array}$$

can be followed by $b \twoheadrightarrow_\beta c$. It is also easy to see that $\mathfrak{A}'$ is reachable, but $\mathfrak{A} \models \mathsf{SN}(a)$, while $\mathfrak{A}'$ has the infinite reduction $b \to_\beta b \to_\beta b \to_\beta \cdots$.

However, ω-simulation does preserve weak termination in both directions:

Theorem 4.7 (Weak Termination)
If $\mathfrak{A} \lesssim^{\phi} \mathfrak{B}$, then we have for all $a \in A$: $\mathfrak{A} \models \mathsf{WN}(a) \iff \mathfrak{B} \models \mathsf{WN}(\phi(a))$.

Proof

($\Rightarrow$) Suppose that $\mathfrak{A} \models \mathsf{WN}(a)$ for some $a \in A$. Then $a \twoheadrightarrow_\alpha n$, where n is some normal form, and thus $\phi(a) \twoheadrightarrow_\beta \phi(n)$ (by the first condition of Definition 3.6 and induction) and $\phi(n) \in \mathsf{NF}(\mathfrak{B})$ (by Proposition 4.4(i)). Thus, $\mathfrak{B} \models \mathsf{WN}(\phi(a))$.

($\Leftarrow$) If $\mathfrak{B} \models \mathsf{WN}(\phi(a))$, then $\phi(a) \twoheadrightarrow_\beta n$ for some $n \in \mathsf{NF}(\mathfrak{B})$. By Proposition 4.4(i–ii) $a \twoheadrightarrow_\alpha \phi^{-1}(n)$ and $\phi^{-1}(n) \in \mathsf{NF}(\mathfrak{A})$, so $\mathfrak{A} \models \mathsf{WN}(a)$. □

Moreover, note that if $\mathfrak{A} \lesssim^{\phi} \mathfrak{B}$ then by the second condition of Definition 3.6 every reachable element reduces to an element in the range of ϕ, so we get:

Corollary 4.8 If $\mathfrak{A} \lesssim^{\phi} \mathfrak{B}$, then we have: $\mathfrak{A} \models \mathsf{WN} \iff \mathfrak{B} \models \mathsf{WN}$. □

4.5 Correctness of Transformations

Recall that Fokkink and Van de Pol [FP96] call a transformation *correct* if it preserves both the normal form relation and termination. They called a transformation *weakly correct* if it weakly preserved the normal form relation and preserved both weak termination and termination.

Note that by Theorem 4.7 ω-simulation preserves weak termination. However, as Example 4.4 demonstrates, it does not preserve termination, so according to Fokkink and Van de Pol, ω-simulation is not a sufficient correctness criterion. It does preserve the normal form relation (Corollary 4.5).

Theorem 4.9 If ϕ is a simulation of $\mathfrak{A}$ by $\mathfrak{B}$, then:

$$(\mathfrak{A} \lesssim^{\phi} \mathfrak{B} + \phi \text{ preserves termination}) \Longrightarrow \mathfrak{B} \text{ is a correct transformation of } \mathfrak{A} \qquad \square$$

Note furthermore that ω-simulation in combination with preservation of termination is essentially stronger then the transformation correctness of [FP96]. This follows from the fact that correctness does not imply preservation of confluence (see Example 4.1), while ω-simulation does (Theorem 4.6). The following example is to emphasize that this not only comes from the fact that in Example 4.1 a simulation map with $\psi \neq \phi^{-1}$ is used:

Example 4.5 Define $\mathfrak{A} = \langle\{a,b\},\{a \to_\alpha a,\ a \to_\alpha b\}\rangle$ and $\mathfrak{B} = \langle\{a,b,c\},\{a \to_\alpha b,\ a \to_\alpha c,\ c \to_\alpha c\}\rangle$. Note that $\phi = \mathsf{id}(\{a,b\})$ preserves termination and the normal form relation, but there is no ω-simulation between these two systems.

5 Term Rewriting Systems

A Term Rewriting System (TRS) $\mathfrak{R}$ is a pair $\langle\Sigma,\mathcal{R}\rangle$ of a signature Σ and a set of reduction rules (rewrite rules) $\mathcal{R}$. A signature is a nonempty set of *function symbols* (we treat constant symbols as nullary function symbols).

We fix a countably infinite set $\mathcal{V}$ of variables, denoted by $x_1, x_2, x_3, \ldots$ or x, y, z.

5.1 Terms

The set $T_{\Sigma,\mathcal{V}}$ of terms over Σ in variables from $\mathcal{V}$ is defined as usual. Terms that contain each variable $x \in \mathcal{V}$ at most once are called *linear*. To make explicit that a term contains no variables other than $x_1, \ldots, x_n$ we will sometimes write $t(x_1, \ldots, x_n)$.

Contexts are 'terms' containing an occurrence of a special symbol $\Box$, denoting an empty place. A context is generally denoted by $C[\,]$. If $t \in T_{\Sigma,\mathcal{V}}$ and t is substituted for $\Box$, then the result is $C[t] \in T_{\Sigma,\mathcal{V}}$.

The notion of a *path* (or *occurrence*, or *position*) is used to refer to subterms in a term t. For any term $t \in T_{\Sigma,\mathcal{V}}$ the set of occurrences $O(t)$ in t is inductively defined as follows:

1. $x \in \mathcal{V} \Longrightarrow O(x) = \{\lambda\}$;
2. $f \in \Sigma$ and $t_1, \ldots, t_n \in T_{\Sigma,\mathcal{V}}$ $(n \geq 0) \Longrightarrow$
   $O(f(t_1, \ldots, t_n)) = \{\lambda\} \cup \{i.u \mid 1 \leq i \leq n \text{ and } u \in O(t_i)\}$.

To the subterm of t reached by p we refer as t/p. The concatenation of p and q (p and q both paths) we denote by $p.q$ so that $t/p.q$ denotes the subterm reached by q in the subterm t/p. We write $o < q$ when there is a nonempty path p such that $o.p = q$; $o \leq q$ when $o < q$ or $o = q$; and we say that o and q are *disjoint* if $o \not\leq q$ and $q \not\leq o$. We moreover divide $O(t)$ into a set $O^{\mathcal{F}}(t)$ of function symbol occurrences and a set $O^{\mathcal{V}}(t)$ of variable occurrences:

$$O^{\mathcal{F}}(t) = \{o \in O(t) \mid t/o \text{ is a nonvariable subterm of } t\}$$

and

$$O^{\mathcal{V}}(t) = \{o \in O(t) \mid t/o \text{ is a variable}\}$$

If $o \in O^{\mathcal{F}}(t)$, then we will denote by $t(o)$ the outermost symbol of t/o, i.e. if $t/o = f(t_1, \ldots, t_n)$, then $t(o) = f$. If o is an occurrence of symbol f in t and $o \neq \lambda$, then o is an *inner* occurrence of f in t.

If σ is a substitution, $t(x_1, \ldots, x_n) \in T_{\Sigma,\mathcal{V}}$ and $\sigma(x_i) = t_i$ for $1 \leq i \leq n$, then $\sigma(t) = t^\sigma = t(t_1, \ldots, t_n)$.

For $s, t \in T_{\Sigma,\mathcal{V}}$ we say that *s overlaps t* if there are substitutions σ, τ and an occurrence $o \in O^{\mathcal{F}}(t)$, such that $\sigma(t/o) = \tau(s)$. Sometimes we need to be explicit about where the overlap takes place. Therefore we also say that *s overlaps t at o*. If $o = \lambda$ then *s root overlaps t* and if $o \neq \lambda$ then *s nonroot overlaps t*.

We extend these notions of overlap to sets of terms. A set $X \subseteq T_{\Sigma,\mathcal{V}}$ is said to be *nonoverlapping* if $\neg \exists s,t \in X$ (s nonroot overlaps t). If $\forall s,t \in X$(s overlaps $t \Longrightarrow s = t$) then X is said to have *no overlap*.

5.2 Rules

A *reduction rule* over Σ is a pair $\langle l, r\rangle$ with $l, r \in T_{\Sigma,\mathcal{V}}$. We usually write $l \to r$ and we assume that $l \notin \mathcal{V}$ and that the variables of r occur in l.

A set $\mathcal{R}$ of rewrite rules generates a reduction relation $\to_{\mathcal{R}}$ on $T_{\Sigma,\mathcal{V}}$ as follows. For $s, t \in T_{\Sigma,\mathcal{V}}$ we write $s \to_{\mathcal{R}} t$ if there is an occurrence $o \in O(s)$, a substitution σ and a rule $l \to r$ in $\mathcal{R}$, such that $s/o = l^\sigma$ and $t = s[o \leftarrow r^\sigma]$. Here $s[o \leftarrow r^\sigma]$ denotes the result of replacing the term s/o in s at occurrence o by r^σ. To be specific about the position in s at which the reduction takes place, we sometimes write $s \xrightarrow{o}_{\mathcal{R}} t$. In case $o = \lambda$, then we call this a *root reduction* (notation: $s \xrightarrow{r}_{\mathcal{R}} t$), while if $o \neq \lambda$ we call it a *nonroot reduction* (notation: $s \xrightarrow{nr}_{\mathcal{R}} t$). A root reduction is also called a *contraction*.

An instance l^σ of a lefthand side l of a rule $l \to r$ in $\mathcal{R}$ is called a *reducible expression* (short: *redex* or l*-redex*).

A *subtemplate* of $\mathcal{R}$ is a nonvariable proper subterm of a lefthand side of a rule in $\mathcal{R}$. An instance of a subtemplate l' of $\mathcal{R}$ is called an l'*-pseudo-redex*, or *pseudo-redex* for short. If s is an l'-pseudo-redex and $s \xrightarrow{o}_{\mathcal{R}} t$ with $o \notin O^{\mathcal{F}}(l')$, then we write $s \xrightarrow[l']{nt}_{\mathcal{R}} t$; we drop the subscript l' if the specific subtemplate is not important.

Let ρ be a reduction, say $s \xrightarrow{o_1}_{\mathcal{R}} s_1 \xrightarrow{o_2}_{\mathcal{R}} \cdots \xrightarrow{o_n}_{\mathcal{R}} s_n$ $(n \geq 1)$, and $o \in O(s)$. We define the *set of descendants* $\mathsf{desc}(o,\rho)$ *of o over ρ* as follows:

1. If $n = 1$ and $s = C[l(t_1, \ldots, t_m)] \xrightarrow{o_1}_{\mathcal{R}} C[r(t'_1, \ldots, t'_m)]$, then

$$\mathsf{desc}(o,\rho) = \begin{cases} \emptyset & \text{if } o = o_1.p \text{ and } p \in O^{\mathcal{F}}(l) \\ \{o_1.p'.q \mid r(p') = x_i\} & \text{if } o = o_1.p.q \text{ and } l/p = x_i \\ \{o_1\} & \text{otherwise} \end{cases}$$

   Note that there may be more than one rule $l \to r$ that applies. If this is the case, then we take the union of the sets of descendants we get via the respective rules.

2. If $n > 1$, and ρ' is the reduction $s \xrightarrow{o_1}_{\mathcal{R}} s_1 \xrightarrow{o_2}_{\mathcal{R}} \cdots \xrightarrow{o_{n-1}}_{\mathcal{R}} s_{n-1}$, then $\mathsf{desc}(o,\rho) = \bigcup_{d \in \mathsf{desc}(o,\rho')} \mathsf{desc}(d, s_{n-1} \xrightarrow{o_n}_{\mathcal{R}} s_n)$.

The *underlying* ARS of $\mathfrak{R}$ is $\langle T_{\Sigma,\mathcal{V}}, \to_{\mathcal{R}}\rangle$. We generally identify $\mathfrak{R}$ with its underlying ARS.

5.3 Subclasses

By imposing (syntactic) restrictions on the rules we can define subclasses of the class of TRSs. For instance, assume a TRS $\mathfrak{R} = \langle \Sigma, \mathcal{R}\rangle$:

- if the base $\mathcal{L}^{\mathcal{R}}$ of $\mathcal{R}$ is linear then $\mathfrak{R}$ is called *left-linear*;
- if the base $\mathcal{L}^{\mathcal{R}}$ of $\mathcal{R}$ is nonoverlapping then $\mathfrak{R}$ is called *nonoverlapping*;
- if $\mathcal{R}$ is nonoverlapping and in addition

$$\forall \langle l, r\rangle, \langle l', r'\rangle \in \mathcal{R}\ \forall \sigma, \tau\ (l^\sigma = (l')^\tau \Longrightarrow r^\sigma = (r')^\tau)$$

  then $\mathfrak{R}$ is called *consistent*;

- if $\mathcal{R}$ is nonoverlapping and in addition the lefthand sides of no two distinct rules overlap at the root, then $\mathfrak{R}$ is called *nonambiguous*;
- if $\forall\langle l,r\rangle,\langle l',r'\rangle\in\mathcal{R}\,\forall\sigma,\tau\;(l^\sigma=(l')^\tau \implies \exists t\in T_{\Sigma,\mathcal{V}}(r^\sigma \twoheadrightarrow_{\mathcal{R}} t \twoheadleftarrow_{\mathcal{R}} (r')^\tau))$, then $\mathfrak{R}$ is said to have *confluent root overlaps*.

Proposition 5.1
nonambiguity $\implies$ consistency $\implies$ nonoverlapping & confluent root overlaps □

Proposition 5.2
Every nonoverlapping TRS with confluent root overlaps is weakly confluent. □

Corollary 5.3 Every nonambiguous (consistent) TRS is weakly confluent.

If a TRS is both nonambiguous and left-linear, then it is called *orthogonal* and if it is both consistent and left-linear, then we call it *regular*.

In term rewriting orthogonal and regular systems are of special importance, mainly because descendants of redexes are still redexes. This leads to the following result:

Corollary 5.4 Every regular TRS is confluent.

Proof See [BKM89] for a proof of confluence for orthogonal TRSs (there called regular). It is easy to verify that their proof generalizes to regular systems. □

6 Thatte's Transformation for Left-Linear Systems

In [Tha85], Satish R. Thatte introduced a transformation for TRSs that turns an arbitrary regular system into a constructor-based system. The characteristic property of these systems is that there signatures may be partitioned into sets of *defined symbols* (appearing only as outermost function symbols of lefthand sides of rules) and *constructor symbols* (not appearing as outermost symbols).

Constructor-based systems are not only interesting for reasons of clarity; regular constructor-based systems are important in the context of strong sequentiality, a property that implies the existence of a normalizing reduction strategy. Strong sequentiality is decidable for regular systems, but the decision procedure is complicated. Thatte in [Tha87] introduces an alternative in the form of 'left sequentiality' for which the decision procedure is much simpler. He proves that for regular constructor-based systems this property coincides with strong sequentiality. Klop and Middeldorp conjecture that this is not the case for regular systems in general [KM91].

6.1 The Transformation

Definition 6.1 Let $\mathfrak{R}=\langle\Sigma,\mathcal{R}\rangle$ be a TRS.
We define the set of *defined symbols* of $\mathfrak{R}$ by

$$\mathcal{F}^{\mathcal{R}}=\{f\in\Sigma\mid\exists t_1,\ldots,t_n\in T_{\Sigma,\mathcal{V}}(f(t_1,\ldots,t_n)\in\mathcal{L}^{\mathcal{R}})\}$$

and the set of *argument symbols* of $\mathfrak{R}$ by

$$\mathcal{A}^{\mathcal{R}}=\{l(o)\mid l\in\mathcal{L}^{\mathcal{R}}\wedge o\in O^{\mathcal{F}}(l)\wedge o\neq\lambda\}$$

Denote by $\mathcal{C}^{\mathcal{R}}$ the set of *constructor symbols* (*constructors*) of $\mathcal{R}$, that is, $\mathcal{C}^{\mathcal{R}}=\Sigma-\mathcal{F}^{\mathcal{R}}$.
$\mathfrak{R}$ is called *constructor-based* if $\mathcal{A}^{\mathcal{R}}\subseteq\mathcal{C}^{\mathcal{R}}$ (or equivalently: $\mathcal{F}^{\mathcal{R}}\cap\mathcal{A}^{\mathcal{R}}=\emptyset$).
If no confusion can arise, then we will drop the superscript $\mathcal{R}$. ⊣

Systems that are not constructor-based have a symbol that is both defined and appears in some argument of a lefthand side, i.e. such a symbol has a dual rôle. The idea is to remove this duality by adding a fresh symbol c_f for each $f \in \mathcal{F} \cap \mathcal{A}$, that will act as the constructor variant of f (take over the rôle of f as a constructor).

Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a TRS. We construct from $\mathfrak{R}$ a system $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp \rangle$ as follows.

We first extend Σ by adding a fresh symbol c_f of arity n (a constructor variant) for each $f \in \mathcal{F} \cap \mathcal{A}$ of arity n:

$$\Sigma^\sharp = \Sigma \cup \{c_f \mid f \in \mathcal{F} \cap \mathcal{A}\}$$

We need translations from Σ-terms to $\Sigma^\sharp$-terms and vice versa. Therefore, we define functions $h : T_{\Sigma^\sharp,\mathcal{V}} \to T_{\Sigma,\mathcal{V}}$ and $c : T_{\Sigma,\mathcal{V}} \to T_{\Sigma^\sharp,\mathcal{V}}$. The function h replaces each occurrence of a constructor variant c_f by its corresponding $f \in \mathcal{F} \cap \mathcal{A}$, while c does the inverse: it replaces each $f \in \mathcal{F} \cap \mathcal{A}$ by its constructor variant c_f. In addition, we define a mapping $ic : T_{\Sigma,\mathcal{V}} \to T_{\Sigma^\sharp,\mathcal{V}}$ that replaces each *inner occurrence* of a symbol $f \in \mathcal{F} \cap \mathcal{A}$ by its constructor variant f, i.e. $ic(f(t_1, \ldots, t_n)) = f(c(t_1), \ldots, c(t_n))$. For convenience we write t^h, t^c and t^{ic} for $h(t)$, $c(t)$ and $ic(t)$, respectively.

We define two sets of rules $\mathcal{R}^\sharp{}_1$ and $\mathcal{R}^\sharp{}_2$.

$\mathcal{R}^\sharp{}_1$ will consist of transformed versions of rules in $\mathcal{R}$: $\mathcal{R}^\sharp{}_1 = \{\langle l^{ic}, r\rangle \mid \langle l, r\rangle \in \mathcal{R}\}$.

In $\mathcal{R}^\sharp{}_2$ we put rules to transform a symbol $f \in \mathcal{F}^\mathcal{R} \cap \mathcal{A}^\mathcal{R}$ into its constructor variant. That is, $\mathcal{R}^\sharp{}_2$ is the smallest set satisfying:

> If u is a subtemplate of a rule in $\mathcal{R}$, with $u = f(t_1, \ldots, t_n)$ and $f \in \mathcal{F}^\mathcal{R} \cap \mathcal{A}^\mathcal{R}$, then $\langle u^{ic}, u^c\rangle \in \mathcal{R}^\sharp{}_2$.

We let $\mathcal{R}^\sharp = \mathcal{R}^\sharp{}_1 \cup \mathcal{R}^\sharp{}_2$. In the sequel we will write $\langle \Sigma^\sharp, \mathcal{R}^\sharp{}_1 \cup \mathcal{R}^\sharp{}_2\rangle$ if we are interested in the TRS with underlying ARS $\langle T_{\Sigma^\sharp,\mathcal{V}}, (\to_\alpha)_{\alpha \in \{\mathcal{R}^\sharp{}_1, \mathcal{R}^\sharp{}_2\}}\rangle$ and $\langle \Sigma^\sharp, \mathcal{R}^\sharp\rangle$ if we mean the TRS with underlying simple ARS $\langle T_{\Sigma^\sharp,\mathcal{V}}, \to_{\mathcal{R}^\sharp}\rangle$. Also, $\mathfrak{R}^\sharp_2 = \langle T_{\Sigma^\sharp,\mathcal{V}}, \to_{\mathcal{R}^\sharp{}_2}\rangle$.

Clearly $\mathfrak{R}^\sharp$ is a constructor-based TRS; and $\mathfrak{R}^\sharp_2$ is complete (i.e. strongly normalizing and confluent).

Thatte proved in [Tha85]:

Lemma 6.2 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a TRS, with transform $\mathfrak{R}^\sharp$. Then:

i. Given $t_1, t_2 \in T_{\Sigma^\sharp,\mathcal{V}}$: $t_1 \to_{\mathcal{R}^\sharp} t_2 \Longrightarrow t_1^h \twoheadrightarrow_\mathcal{R} t_2^h$.

ii. Given $t_1, t_2 \in T_{\Sigma,\mathcal{V}}$: $t_1 \to_\mathcal{R} t_2 \Longrightarrow t_1 \to^+_{\mathcal{R}^\sharp} t_2$.

6.2 Correctness for Nonoverlapping Left-Linear TRSs

If $\mathfrak{R}$ is a nonoverlapping left-linear TRS, then so is $\mathfrak{R}^\sharp$. In particular, a lefthand side of an $\mathcal{R}^\sharp{}_2$-rule cannot overlap an $\mathcal{R}^\sharp{}_1$-lefthand side: nonroot overlaps are impossible because $\mathfrak{R}^\sharp$ is constructor-based, and root overlap would imply overlap in $\mathfrak{R}$.

For a TRS $\mathfrak{R}$, let $\phi_\mathfrak{R}$ be the map $\mathsf{nf}_{\mathfrak{R}^\sharp_2} \restriction T_{\Sigma,\mathcal{V}}$. Since $\mathcal{R}^\sharp{}_2$-rules just replace symbols, and distinct symbols by distinct new symbols, $\phi_\mathfrak{R}$ is clearly injective.

The following lemma expresses that every $\Sigma^\sharp$-term t that is reachable from an element in the range of $\phi_\mathfrak{R}$ is reachable from t^h by an $\mathcal{R}^\sharp{}_2$-reduction.

Lemma 6.3 Let $\mathfrak{R} = \langle \Sigma, \mathcal{V} \rangle$ be a TRS.
If $s \in \mathsf{range}(\phi_\mathfrak{R})$, $t \in T_{\Sigma^\sharp,\mathcal{V}}$, then $s \twoheadrightarrow_{\mathcal{R}^\sharp} t \Longrightarrow t^h \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t$.

Proof By Induction on the length of the reduction $s \twoheadrightarrow_{\mathcal{R}^\sharp} t$.

If $s \to^0_{\mathcal{R}^\sharp} t$, then $t = \mathsf{nf}_{\mathcal{R}^\sharp{}_2}(t^h)$.

Suppose the lemma holds for reductions $s \twoheadrightarrow_{\mathcal{R}^\sharp} t'$ of length $n \in \omega$ and $s \twoheadrightarrow^n_{\mathcal{R}^\sharp} t' \to_{\mathcal{R}^\sharp} t$.

If $t' \to_{\mathcal{R}^\sharp{}_2} t$, then we are done because by induction hypothesis $(t')^h \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t'$ and clearly $t^h = (t')^h$. If $t' \to_{\mathcal{R}^\sharp{}_1} t$, say by application of an $\mathcal{R}^\sharp{}_1$-rule $l \to r$ in a context $C[\,]$ using a substitution σ (i.e. $t' = C[l^\sigma] \to_{\mathcal{R}^\sharp{}_1} C[r^\sigma] = t$), then $C[\,]^h \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} C[\,]$ (since the $\mathcal{R}^\sharp{}_2$-rules are left-linear and have no overlap with $\mathcal{R}^\sharp{}_1$-rules). Moreover, if $l^\sigma = l(x_1, \ldots, x_n)^\sigma = l(t_1, \ldots, t_n)$, then $t_i^h \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t_i$ for all $1 \le i \le n$, since $(t')^h \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t'$. Finally, note that the righthand sides of the $\mathcal{R}^\sharp{}_1$-rules are Σ-terms. □

Lemma 6.4 ($\mathcal{R}^\sharp{}_1$-$\mathcal{R}^\sharp{}_2$ **Strip Lemma**) Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a nonoverlapping left-linear TRS. For all $s, s', t \in T_{\Sigma^\sharp, \mathcal{V}}$: $s' \twoheadleftarrow_{\mathcal{R}^\sharp{}_2} s \to_{\mathcal{R}^\sharp{}_1} t$ implies that there is a $t' \in T_{\Sigma^\sharp, \mathcal{V}}$ such that $s' \to_{\mathcal{R}^\sharp{}_1} t' \twoheadleftarrow_{\mathcal{R}^\sharp{}_2} t$.

Proof First we show that $s' \xleftarrow{o_2}_{\mathcal{R}^\sharp{}_2} s \xrightarrow{o_1}_{\mathcal{R}^\sharp{}_1} t$ implies that there is a $t' \in T_{\Sigma^\sharp, \mathcal{V}}$ such that $s' \to_{\mathcal{R}^\sharp{}_1} t' \twoheadleftarrow_{\mathcal{R}^\sharp{}_2} t$.

We distinguish cases according to the relative positions of o_1 and o_2 in s:

$o_1 = o_2$**:** This is impossible. If $o_1 = o_2$, then $(s/o_1)^h = (s/o_2)^h$ would be both an instance of a proper subterm of a lefthand side of a rule in $\mathcal{R}$ as well an instance of some lefthand side of a rule.

$o_1 < o_2$**:** Let $l \to r$ be the $\mathcal{R}^\sharp{}_1$-rule applied at s/o_1 and $o_2 = o_1.p$. Because $\mathfrak{R}^\sharp$ is nonoverlapping, it must be the case that $p \notin O^{\mathcal{F}}(l)$. Thus, in t we can contract all descendants of o_2 using the same rule as the one applied in $s \to_{\mathcal{R}^\sharp{}_2} s'$, yielding a term u and $t \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} u$. On the other hand, again because $\mathfrak{R}^\sharp$ is nonoverlapping and left-linear, in s' it is still possible to reduce s'/o_1, which clearly results in the same u.

$o_2 < o_1$**:** Similar (but note that o_2 has only one descendant, viz. o_2, and that by construction and by left-linearity of $\mathcal{R}$ a righthand side of an $\mathcal{R}^\sharp{}_2$-rule cannot contain the same variable more than once).

o_1 **and** o_2 **disjoint:** Obvious.

Now the lemma follows by induction on the length of the reduction $s \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} s'$. □

Theorem 6.5
If $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ is a nonoverlapping left-linear TRS, then $\phi_{\mathfrak{R}}$ is an ω-simulation of $\mathfrak{R}$ by $\mathfrak{R}^\sharp$.

Proof We prove that the two conditions of Definition 3.6 hold.

First, assume $s \to_{\mathcal{R}} t$ for $s, t \in T_{\Sigma, \mathcal{V}}$. We have to show that $\phi_{\mathfrak{R}}(s) \to^+_{\mathcal{R}^\sharp} \phi_{\mathfrak{R}}(t)$.

If $s \to_{\mathcal{R}} t$ then there are an $o \in O(s)$, a substitution σ, and an $\mathcal{R}$-rule $l \to r$, such that $s/o = l^\sigma$ and $t = s[o \leftarrow r^\sigma]$. Furthermore, $\mathfrak{R}^\sharp$ is constructed in such a way that $s/o = l^\sigma \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} (l^{ic})^\sigma$, and because $\langle l^{ic}, r \rangle \in \mathcal{R}^\sharp{}_1$ we find $s \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} s[o \leftarrow (l^{ic})^\sigma] \to_{\mathcal{R}^\sharp{}_1} s[o \leftarrow r^\sigma] = t$.

Now $s[o \leftarrow (l^{ic})^\sigma] \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} \phi_{\mathfrak{R}}(s)$, so by Lemma 6.4 there is some $t' \in T_{\Sigma^\sharp, \mathcal{V}}$ such that $\phi_{\mathfrak{R}}(s) \to_{\mathcal{R}^\sharp{}_1} t'$ and $t \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t'$. Clearly, both $t \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} \phi_{\mathfrak{R}}(t)$ and $t' \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} \phi_{\mathfrak{R}}(t)$, so $\phi_{\mathfrak{R}}(s) \to^+_{\mathcal{R}^\sharp} \phi_{\mathfrak{R}}(t)$. The following diagram illustrates the procedure just described:

$$\begin{tikzcd}
s \arrow[r, two heads, "\mathcal{R}^\sharp{}_2"] \arrow[d, two heads, "\mathcal{R}^\sharp{}_2"'] & s[o \leftarrow (l^{ic})^\sigma] \arrow[r, "\mathcal{R}^\sharp{}_1"] \arrow[d, two heads, "\mathcal{R}^\sharp{}_2"'] \arrow[dr, phantom, "\text{Lem. 6.4}"] & t \arrow[r, equal] \arrow[d, two heads, "\mathcal{R}^\sharp{}_2"] & t \arrow[d, two heads, "\mathcal{R}^\sharp{}_2"] \\
\phi(s) \arrow[r, equal] & \phi(s) \arrow[r, "\mathcal{R}^\sharp{}_1"'] & t' \arrow[r, two heads, "\mathcal{R}^\sharp{}_2"'] & \phi(t)
\end{tikzcd}$$

Secondly, assume $s \to^+_{\mathcal{R}^\sharp} t$ for some $s \in \mathsf{range}(\phi_{\mathfrak{R}})$ and $t \in T_{\Sigma^\sharp, \mathcal{V}}$. According to Definition 3.6 we have to show the existence of a term $t' \in \mathsf{range}(\phi_{\mathfrak{R}})$ such that $\phi_{\mathfrak{R}}^{-1}(s) \to^+_{\mathcal{R}} \phi_{\mathfrak{R}}^{-1}(t')$ and $t \twoheadrightarrow_{\mathcal{R}^\sharp} t'$.

By Lemma 6.3 $t^h \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t$. Moreover, because $\mathcal{R}^\sharp{}_2$ is complete, and because $\phi_{\mathfrak{R}}(t^h)$ is obtained from t^h by an $\mathcal{R}^\sharp{}_2$-reduction we find $t \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} \phi_{\mathfrak{R}}(t^h)$. So we take $t' = \phi_{\mathfrak{R}}(t^h)$ and it remains to show that $\phi_{\mathfrak{R}}^{-1}(s) = s^h \rightarrow_{\mathcal{R}}^+ t^h = \phi_{\mathfrak{R}}^{-1}(\phi_{\mathfrak{R}}(t^h))$.
Since s is an $\mathcal{R}^\sharp{}_2$-normal form, the first step of the reduction to t must be an application of a $\mathcal{R}^\sharp{}_1$-rule, so we have $s \rightarrow_{\mathcal{R}^\sharp{}_1} s' \twoheadrightarrow_{\mathcal{R}^\sharp} t$. Furthermore, $s \rightarrow_{\mathcal{R}^\sharp{}_1} s'$ implies $s^h \rightarrow_{\mathcal{R}} (s')^h$. So by Lemma 6.2(i) and induction w.r.t. the length of the reduction $s' \twoheadrightarrow_{\mathcal{R}^\sharp} t$, we get $(s')^h \twoheadrightarrow_{\mathcal{R}} t^h$. Therefore $s^h \rightarrow_{\mathcal{R}} (s')^h \twoheadrightarrow_{\mathcal{R}} t^h$, i.e. $\phi_{\mathfrak{R}}^{-1}(s) \rightarrow_{\mathcal{R}}^+ t^h$. □

Theorem 6.6 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a TRS.
The simulations $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h \rangle$ and $\phi_{\mathfrak{R}}$ preserve termination.

Proof Observe that (1) $t \rightarrow_{\mathcal{R}^\sharp{}_1} t'$ implies $t^h \rightarrow_{\mathcal{R}} (t')^h$; and (2) $t \rightarrow_{\mathcal{R}^\sharp{}_2} t'$ implies $t^h = (t')^h$. Now, assume $s \in T_{\Sigma,\mathcal{V}}$ is terminating in $\mathfrak{R}$, while s has an infinite reduction in $\mathfrak{R}^\sharp$. Because $\langle \Sigma^\sharp, \mathcal{R}^\sharp{}_2 \rangle$ is SN and $T_{\Sigma^\sharp,\mathcal{V}}$ contains only finite terms, there must be infinitely many $\mathcal{R}^\sharp{}_1$-steps in this reduction, so it can be written as follows:

$$s \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} s_0 \rightarrow_{\mathcal{R}^\sharp{}_1} s_1 \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} {s_1}' \rightarrow_{\mathcal{R}^\sharp{}_1} \cdots \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} {s_{n-1}}' \rightarrow_{\mathcal{R}^\sharp{}_1} s_n \rightarrow_{\mathcal{R}^\sharp{}_2} \cdots$$

However, because of (1) and (2) together, this corresponds to

$$s = s_0^h \rightarrow_{\mathcal{R}} s_1^h \rightarrow_{\mathcal{R}} \cdots \rightarrow_{\mathcal{R}} s_n^h \rightarrow_{\mathcal{R}}^+ \cdots$$

which contradicts the assumption that s was terminating in $\mathfrak{R}$. So $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h \rangle$ preserves termination, and since $s \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} \phi_{\mathfrak{R}}(s)$ $\phi_{\mathfrak{R}}$ preserves termination as well. □

From Theorems 6.5 and 6.6 we get by Theorem 4.9 that if $\mathfrak{R}$ is a nonoverlapping and left-linear TRS, then $\mathfrak{R}^\sharp$ is a correct transformation of $\mathfrak{R}$ as in [FP96].

7 Enlarging the Domain of Thatte's Transformation

In 1988, Thatte tried to extend his result to a larger class of nonoverlapping confluent systems (without the restriction of left-linearity) [Tha88]. He based his correctness argument on the presumed fact that for the given class of rewrite systems his transformation resulted in a system that was confluent for reachable terms.

Unfortunately, Thatte's proof for the class of nonoverlapping confluent systems does not work, and a counterexample of Rakesh M. Verma [Ver95b] shows that his theorem does not even hold. Inspired by his counterexample, Verma somewhat restricted the domain of the transformation and stated the theorem that for this domain the transformation preserved confluence.

However, we discovered some fundamental errors in Verma's proof (see appendix A), and after we had notified him of these, he discovered a counterexample for his theorem [Ver96]. In this section we will discuss and simplify the restriction of the domain for Thatte's transformation that Verma proposed in [Ver95b]. We conclude with some positive preservation results.

7.1 Weak Persistence

In [Tha88] Thatte stated the following as a theorem:

Claim 7.1 ([Tha88])
If $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ is nonoverlapping and confluent, then $\mathfrak{R}^\sharp[T_{\Sigma,\mathcal{V}}]$ is confluent.

This is clearly the same as stating that the simulation mapping $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h \rangle$ (h as defined in the previous section) preserves confluence. Verma refutes this claim. We will repeat his counterexample and show that it also works for the mapping $\phi_{\mathfrak{R}}$ that we used in the previous section. In appendix B we give the nontrivial proof that the original TRS of Verma's counterexample is confluent.

From his counterexample, Verma derived an extra restriction that is necessary to obtain a confluence preserving simulation map. We will define this restriction (Verma called it 'weak persistence') at the end of this section.

Example 7.1 ([Ver95b]) Consider the TRS $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ and its transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp \rangle$, where $\Sigma = \{f, g, h, a, c\}$ and $\Sigma^\sharp = \Sigma \cup \{c_f\}$. $\mathcal{R}$ and $\mathcal{R}^\sharp$ respectively consist of the rules:

$$\mathcal{R} = \left\{ \begin{array}{lcl} h(f(x,x)) & \to & h(a), \\ f(x,g(x)) & \to & a, \\ c & \to & g(c) \end{array} \right\}, \quad \mathcal{R}^\sharp = \left\{ \begin{array}{lcl} h(c_f(x,x)) & \to & h(a), \\ f(x,g(x)) & \to & a, \\ c & \to & g(c), \\ f(x,x) & \to & c_f(x,x) \end{array} \right\}$$

In Lemma B.1 (see appendix B) we show that $\mathfrak{R}$ is indeed confluent. However, $\mathfrak{R}^\sharp$ is not confluent for reachable terms. Consider the Σ-term $f(g(c), c)$.[3] Note that it is a ${\mathcal{R}^\sharp}_2$-normal form, so under $(\mathsf{id}(T_{\Sigma,\mathcal{V}}), h)$ as well as under $\phi_{\mathfrak{R}}$ this term is simulated in $\mathfrak{R}^\sharp$ by $f(g(c), c)$. Furthermore, $f(g(c), c) \to_{{\mathcal{R}^\sharp}_1} f(g(c), g(c))$ so $f(g(c), g(c))$ is a reachable term under both simulation maps. However, $\mathfrak{R}^\sharp \models \mathsf{CR}(f(g(c), g(c)))$ does not hold (see the diagram below).

$$\begin{tikzcd} f(g(c),g(c)) \arrow[r, "{\mathcal{R}^\sharp}_1"'] \arrow[d, "{\mathcal{R}^\sharp}_2"] & f(g(c),g(g(c))) \arrow[r, "{\mathcal{R}^\sharp}_1"'] & a \\ c_f(g(c),g(c)) \arrow[d, dashed, "{\mathcal{R}^\sharp}_1", "/"'] & & \\ {} & & \end{tikzcd}$$

It is clear that $c_f(g(c), g(c))$ never reduces to a, because there is no rule that removes the c_f-symbol from this term.

The problem in the above counterexample lies in the fact that $\mathfrak{R}$ (although it is a nonoverlapping TRS) contains some ambiguity. For instance, for the Σ-term $h(f(g(c), g(c)))$, we have two different kinds of reductions to $h(a)$:

1. the reductions using $h(f(x,x)) \to h(a)$, e.g. $h(f(g(c), g(c)) \to_{\mathcal{R}} h(a)$; and
2. the reductions using $f(x, g(x)) \to a$, e.g. $h(f(g(c), g(c)) \to_{\mathcal{R}} h(f(g(c), g(g(c)))) \to_{\mathcal{R}} h(a)$.

Note that in the reductions of the first kind f acts as a constructor symbol, while in the reductions of the second kind it acts as a defined symbol. Intuitively, the first step in the second reduction, that is, the application of the rule $c \to g(c)$ to the second argument of f, switches the rôle of f from constructor to defined symbol. Note that this switch is not rigid, because if we apply $c \to g(c)$ to the first argument of f in $h(f(g(c), g(g(c)))$ the rule $h(f(x,x)) \to h(a)$ is applicable again.

In $\mathfrak{R}^\sharp$ things are a bit different. We still have this switching capability, but now we also have the possibility of making the switch rigid. In the situation where f appears in its rôle of constructor (say in terms $f(t,t)$) we can now make that situation explicit by applying the rule $f(x,x) \to c_f(x,x)$, and then there is no return.

[3] Verma uses the term $f(c, c)$. This term works for the mapping $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h \rangle$, but not for our mapping $\phi_{\mathfrak{R}}$.

To avoid this situation, Verma introduces his 'weak persistence'. Informally, this notion makes sure that instances of terms in which the outermost symbol acts as a constructor symbol (instances of proper subterms of lefthand sides of rules) do not reduce to instances of lefthand sides of rules.

Definition 7.2 (Weak Persistence) Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a TRS.

We say that $\mathfrak{R}$ is *weakly persistent* if there is no subtemplate s of $\mathcal{R}$ and s-pseudo-redex $t \in T_{\Sigma,\mathcal{V}}$ such that $\exists t_1, t_2 \in T_{\Sigma,\mathcal{V}} (t \xrightarrow[s]{nt}{}^{*}_{\mathcal{R}} t_1 \wedge t_1 \xrightarrow{r}_{\mathcal{R}} t_2)$. ⊣

We refer to section 5.2 for definitions of 'subtemplate', 'pseudo-redex', $\xrightarrow{nt}_{\mathcal{R}}$ and $\xrightarrow{r}_{\mathcal{R}}$.

Note that $\mathfrak{R}$ of Example 7.1 is not weakly persistent, because the switching reductions are the nontemplate reductions of the above definition.

The above formulation of weak persistence differs from that of Verma, but defines an equivalent notion:

Lemma 7.3 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a TRS. The following statements are equivalent:

1. $\mathfrak{R}$ is weakly persistent as in [Ver95b].

2. $\mathfrak{R}$ is weakly persistent as defined in Definition 7.2.

Proof

(1 ⇒ 2) Suppose $\mathfrak{R}$ is weakly persistent as in [Ver95b]; we show that the conditions of Definition 7.2 are met. Let s be a subtemplate of $\mathcal{R}$ and t an s-pseudo-redex. Moreover, let $t \xrightarrow[s]{nt}{}^{*}_{\mathcal{R}} t_1$ for some $t_1 \in T_{\Sigma,\mathcal{V}}$. Clearly, the root of t is untouched in this reduction (i.e. $t \xrightarrow{nr}{}^{*}_{\mathcal{R}} t_1$) and thus it cannot be the case that $t_1 \xrightarrow{r}_{\mathcal{R}} t_2$ for some $t_2 \in T_{\Sigma,\mathcal{V}}$, for this would contradict weak persistence according to Verma.

(2 ⇒ 1) Contraposition. Suppose s is a subtemplate and t is an s-pseudo-redex contradicting the weak persistence of $\mathfrak{R}$ as in [Ver95b], i.e. there are a $t_1 \in T_\Sigma$ such that $t \xrightarrow{nr}{}^{*}_{\mathcal{R}} t_1$ and an $o \in O(t_1) \cap O^{\mathcal{F}}(s)$ with t_1/o a redex. We show that t/o is an s/o-pseudo-redex that contradicts weak persistence as in Definition 7.2 of $\mathfrak{R}$.

First observe that without loss of generality we may assume that $t \xrightarrow[s]{nt}{}^{*}_{\mathcal{R}} t_1$. If $t = t_1$ this is clear and if $t = t'_0 \xrightarrow{nr}_{\mathcal{R}} t'_1 \xrightarrow{nr}_{\mathcal{R}} \cdots \xrightarrow{nr}_{\mathcal{R}} t'_n = t_1$ $(n \geq 1)$, then the the first t'_{i+1} $(0 < i < n)$ such that $t'_i \xrightarrow[s]{nt}_{\mathcal{R}} t'_{i+1}$ does not hold, would already contradict weak persistence according to Verma, so we could have taken $t_1 = t'_i$ and $o = o'$, for t'_i/o' the contracted redex of $t'_i \xrightarrow{nr}_{\mathcal{R}} t'_{i+1}$.

Furthermore, observe that s/o is a subtemplate of $\mathcal{R}$, so t/o is an s/o-pseudo-redex, $t/o \xrightarrow[s/o]{nt}{}^{*}_{R} t_1/o$ and there is some rule of $\mathcal{R}$ applicable at the root of t_1/o.

Thus, we conclude that the existence of a pseudo-redex contradicting weak persistence according to Verma implies the existence of a a pseudo-redex contradicting weak persistence as in Definition 7.2. □

7.2 Conditions for Weak Persistence

Verma states the theorem that each subclass of TRSs that meets some requirements contains only weakly persistent systems. As corollaries of this, Verma obstains that the following systems are weakly persistent:

1. left-linear nonoverlapping systems with confluent root overlaps;

2. nonoverlapping strongly normalizing systems with confluent root overlaps; and

3. the subclass consisting of systems $\langle \Sigma \cup \{D_i, E\}, \mathcal{R} \cup \mathcal{D}_i \rangle$ ($i = k$ or $i = s$), where it is assumed that $D_k, D_s, E \notin \Sigma$, $\langle \Sigma, \mathcal{R} \rangle$ is a regular TRS, $\mathcal{D}_s = \{D_s(x,x) \to E\}$, and $\mathcal{D}_k = \{D_k(x,x) \to E(x)\}$ (these systems appear in [Klo80]).

In this section we will give proofs of the above mentioned theorem, using our own definition, and for Verma's corollaries.

To show the weak persistence of the first type of systems mentioned above we use the following proposition (instead of Verma's theorem).

Proposition 7.4 i. Every weakly persistent TRS is nonoverlapping.

ii. Every left-linear nonoverlapping TRS is weakly persistent.

Proof

i. Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a weakly persistent TRS, and assume that $l, l' \in \mathcal{L}^{\mathcal{R}}$, $o \in O(l)$, $o \neq \lambda$, and σ is a substitution such that $\sigma(l/o) = \sigma(l')$. Obviously, l/o is a subtemplate of $\mathcal{R}$, so $\sigma(l/o)$ is both a l/o-pseudo-redex and a l'-redex. This contradicts weak persistence.

ii. Suppose $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ is not weakly persistent. Then there is a substitution and some proper subterm s (say at position $o \in O(l)$) of lefthand side l of some rule in $\mathcal{R}$, such that $s^\sigma \xrightarrow[s]{nt}{}^*_{\mathcal{R}} s'$ and $s' \xrightarrow{r}_{\mathcal{R}} t$. But clearly, because s is a linear term, there must be some substitution τ, such that $s' = s^\tau$. Now $s^\tau \xrightarrow{r}_{\mathcal{R}} t$ implies that there is some rule $(l', r) \in \mathcal{R}$ and a substitution ρ, such that $(l')^\rho = s^\tau = (l/o)^\tau$. Thus, there is overlap and $\mathfrak{R}$ is not nonoverlapping. □

Now it is trivial by the above proposition that left-linear nonoverlapping systems with confluent root overlaps (systems of the first type) are weakly persistent.

Theorem 7.5 (Theorem 9 of [Ver95b]) Let $\mathcal{C}$ be a class of nonoverlapping, uniquely normalizing TRSs $\langle \Sigma, \mathcal{R} \rangle$ closed under name permutations and extensions $\mathcal{R} := \mathcal{R} \cup \{l \to a\}$ ($l \in T_{\Sigma,\mathcal{V}}$, a a new constant symbol, i.e. $a \notin \Sigma$) such that:

- $\{l\}$ is nonoverlapping;
- for all nonvariable subterms t_1 of a lefthand side m of a rule in $\mathcal{R}$ and t_2 of l:

$$\neg\, \exists \sigma, \tau\ (\sigma(t_1) = \tau(l) \vee \sigma(t_2) = \tau(m))$$

Then $\mathcal{C}$ only contains weakly persistent systems.

Proof Assume that $\mathcal{C}$ is as above, $\mathfrak{R} \in \mathcal{C}$ and $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ is *not* weakly persistent.

There must be a subtemplate s of some lefthand side l of a rule in $\mathcal{R}$ and an s-pseudo-redex t, such that $t \xrightarrow[s]{nt}_{\mathcal{R}} t_1$, with t_1 a redex for a rule $l_1 \to r_1$ in $\mathcal{R}$ ($t = \tau(l_1)$, for substitution τ).

Assume that $q, d, c \notin \Sigma \cup \underline{\Sigma}$ and define $\mathfrak{G} = \langle \Gamma, \mathcal{G} \rangle$ by

$$\mathfrak{G} := \langle \Sigma \cup \underline{\Sigma} \cup \{q, d, e\}, \underline{\mathcal{R}} \cup \{q(s') \to d, l_1' \to e\} \rangle$$

in which s' and l' are obtained by underlining function symbols at all occurrences except the root.

It is easy to verify that the extension is such that it implies that $\mathfrak{G} \in \mathcal{C}$ (note that if l_1' overlaps s', then l_1 overlaps s, quod non).

Now, to arrive at a contradiction, we prove that $\mathfrak{G}$ is not uniquely normalizing. Let $t = \sigma(s)$; $u_1 = q(\underline{\sigma}(s'))$. By our choice of u_1 and construction of $\mathfrak{G}$, there is a term u_2 corresponding to t_1, such that $u_1 \xrightarrow[q(s')]{nt}_{\underline{\mathcal{R}}} q(u_2)$ and u_2 is an instance of l_1'. Now $u_1 \xrightarrow{r}_{\mathcal{G}} d$ via $q(s') \to d$ and $q(u_2) \to_{\mathcal{G}} q(e)$. Clearly, both d and $q(e)$ are normal forms, so $\mathfrak{G}$ is not uniquely normalizing. We conclude that there cannot be a $\mathcal{C}$ satisfying all conditions mentioned that contains a system that is not weakly persistent. □

Note that the requirement that classes as meant in the above theorem should be closed under extensions $\{l \to a\}$ in which a is a constant, leads to an essentially stronger theorem, than if the requirement on a had been that is is just an arbitrary normal form.

Furthermore, Verma included in his formulation of the theorem the requirement that l should be linear if all $\mathfrak{R} \in \mathcal{C}$ are left-linear. This requirement does not play a rôle in the proof and is only to make the theorem also applicable to subclasses of the class of left-linear, nonoverlapping systems (these classes trivially are not closed under extension with a nonlinear rule). However, Proposition 7.4(ii) already states that the whole class of left-linear, nonoverlapping systems only contains weakly persistent systems.

As corollaries of the above theorem we now get that nonoverlapping strongly normalizing TRSs as well as some of the nonlinear systems mentioned in [Klo80] are weakly persistent.

Corollary 7.6 Nonoverlapping, strongly normalizing TRSs with confluent root overlaps are weakly persistent.

Proof We show that the class of nonoverlapping, strongly normalizing TRSs with confluent root overlaps meets the conditions of Theorem 7.5.

Note that if $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ is a nonoverlapping TRS with confluent root overlaps, then $\mathfrak{R} \models \mathsf{WCR}$ by Proposition 5.2. In addition $\mathfrak{R} \models \mathsf{SN}$ by assumption, so by an application of Newman's Lemma $\mathfrak{R} \models \mathsf{CR}$. Thus, we get that $\mathfrak{R} \models \mathsf{UN}$.

Moreover, if a rule $l \to a$ is added ($\mathcal{R}^+ = \mathcal{R} \cup \{l \to a\}$), such that l has no overlap with a rule in $\mathcal{R}$, then (trivially) the resulting system is nonoverlapping and has confluent root overlaps.

Finally, the resulting system is still strongly normalizing, for if it has an infinite reduction

$$s_0 \to_{\mathcal{R}^+} s_1 \to_{\mathcal{R}^+} \cdots \to_{\mathcal{R}^+} s_n \to_{\mathcal{R}^+} \cdots \qquad (s_i \in T_{\Sigma \cup \{a\}, \mathcal{V}},\ i \in \omega)$$

then we construct an infinite reduction in $\mathfrak{R}$ as follows (and the result follows by contraposition).

Notice that in the above reduction there must be infinitely many applications of $\mathcal{R}$-rules. Let $\chi : T_{\Sigma \cup \{a\}, \mathcal{V}} \to T_{\Sigma, \mathcal{V}}$ be the function that replaces in $t \in T_{\Sigma \cup \{a\}, \mathcal{V}}$ each occurrence of the symbol a by the $T_{\Sigma, \mathcal{V}}$-term l. If for $s, t \in T_{\Sigma \cup \{a\}, \mathcal{V}}$ and $s \to_{\mathcal{R}} t$, then $\chi(s) \to_{\mathcal{R}} \chi(t)$. Moreover, if $s \to_{\mathcal{R}^+} t$ by an application of the rule $l \to a$, then $\chi(s) = \chi(t)$.

Therefore, the above infinite reduction translates to

$$\chi(s_0) \twoheadrightarrow_{\mathcal{R}} \chi(s_1) \twoheadrightarrow_{\mathcal{R}} \cdots \to_{\mathcal{R}} \chi(s_n) \to_{\mathcal{R}} \cdots \qquad (s_i \in T_{\Sigma \cup \{a\}, \mathcal{V}},\ i \in \omega)$$

and there are infinitely many $i \in \omega$ such that $s_i \to_{\mathcal{R}} s_{i+1}$. □

Corollary 7.7 The subclass of systems $\langle \Sigma \cup \{D_i, E\}, \mathcal{R} \cup \mathcal{D}_i \rangle$ ($i = k$ or $i = s$), where it is assumed that $D_k, D_s, E \notin \Sigma$, $\langle \Sigma, \mathcal{R} \rangle$ is a regular TRS, $\mathcal{D}_s = \{D_s(x,x) \to E\}$, and $\mathcal{D}_k = \{D_k(x,x) \to E(x)\}$ contains only weakly persistent TRSs.

Proof Note that a regular system extended in the way Theorem 7.5 describes is still regular.

Furthermore, Klop proves in [Klo80] that the mentioned systems are confluent, so they are uniquely normalizing as well. □

7.3 Positive Results

In this section we collect a number of positive preservation results that we obtained for Thatte's transformation. In particular, we show that for the systems that we proved weakly persistent in the previous section Thatte's transformation preserves confluence. For most of these systems we even get ω-simulations.

According to a counterexample due to Verma [Ver96], weak persistence is not a sufficient condition to obtain that Thatte's transformation preserves confluence. We will show, however, that it is sufficient to guarantee that Thatte's transformation preserves the unique normal form property and weak termination. As a corollary of this, we obtain that Thatte's transformation is correct as in [FP96] for uniquely normalizing, weakly persistent systems and that it preserves semi-completeness for weakly persistent systems.

First, recall that systems of the first type mentioned in the previous section are in the scope of section 6; the present section deals with the other two.

Definition 7.8 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a TRS.

We say that $\mathfrak{R}$ has *linear constructor variants* if all subtemplates s with outermost symbol from $\mathcal{F} \cap \mathcal{A}$ (i.e. $s = f(t_1, \ldots, t_n)$ where $f \in \mathcal{F} \cap \mathcal{A}$, $t_1, \ldots, t_n \in T_{\Sigma,\mathcal{V}}$) are linear. $\dashv$

Observe that if $\mathfrak{R}$ is a nonoverlapping TRS with linear constructor variants, then $\langle \Sigma^\sharp, \mathcal{R}^\sharp{}_2 \rangle$ is still left-linear. Therefore, Lemma 6.3 still holds for this class of systems (the proof of this lemma only uses the left-linearity of the $\mathcal{R}^\sharp{}_2$-rules). Unfortunately, Lemma 6.4 does not hold for this class. Below, we will prove a weaker version, which will turn out to be enough. In the proof of this weaker version we need the following proposition.

Proposition 7.9 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a nonoverlapping TRS with linear constructor variants.

For all $s, s', t \in T_{\Sigma^\sharp,\mathcal{V}}$: if $s' \leftarrow_{\mathcal{R}^\sharp{}_2} s \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t$, then there is a $t' \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $s' \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t'$, $t \to^i_{\mathcal{R}^\sharp{}_2} t'$ and $i \in \{0, 1\}$.

Proof Induction w.r.t. the length of the reduction $s \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t$.

The case where $s \to^0_{\mathcal{R}^\sharp{}_2} t$ is trivial: take $t' = s'$. If $s \to^n_{\mathcal{R}^\sharp{}_2} u \to_{\mathcal{R}^\sharp{}_2} t$ and the lemma holds for reductions of length n, then the induction hypothesis yields a $u' \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $s' \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} u'$ and $u \to^i_{\mathcal{R}^\sharp{}_2} u'$ ($i \in \{0, 1\}$). If $i = 0$, then $u = u'$ and thus $s \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} u \to_{\mathcal{R}^\sharp{}_2} t$, so we are done.

If $i = 1$ then assume that $u \xrightarrow{o_1}_{\mathcal{R}^\sharp{}_2} t$ and $u \xrightarrow{o_2}_{\mathcal{R}^\sharp{}_2} u'$. If $o_1 = o_2$, then we take $t' = u' = t$, and if o_1 and o_2 are disjoint, then in t, u' we contract the single descendant of o_1, o_2 respectively yielding $u' \to_{\mathcal{R}^\sharp{}_2} t' \leftarrow_{\mathcal{R}^\sharp{}_2} t$. If $o_1 < o_2$, or $o_2 < o_1$, then since all rules in $\mathcal{R}^\sharp{}_2$ are left-linear by assumption, so t/o_2 and u'/o_1 are again $\mathcal{R}^\sharp{}_2$-redexes. Contraction of these redexes yields $u' \to_{\mathcal{R}^\sharp{}_2} t' \leftarrow_{\mathcal{R}^\sharp{}_2} t$. $\square$

Lemma 7.10 (Weak $\mathcal{R}^\sharp{}_1$-$\mathcal{R}^\sharp{}_2$ Strip Lemma) Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a nonoverlapping TRS with linear constructor variants.

For all $s, s', t \in T_{\Sigma^\sharp,\mathcal{V}}$: if $s' \twoheadleftarrow_{\mathcal{R}^\sharp{}_2} s \to_{\mathcal{R}^\sharp{}_1} t$, then there are $t', t'' \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $s' \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t' \to_{\mathcal{R}^\sharp{}_1} t'' \twoheadleftarrow_{\mathcal{R}^\sharp{}_2} t$.

Proof We first show that $s' \xleftarrow{o_2}_{\mathcal{R}^\sharp{}_2} s \xrightarrow{o_1}_{\mathcal{R}^\sharp{}_1} t$ implies that there are $t', t'' \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $s' \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t' \to_{\mathcal{R}^\sharp{}_1} t'' \twoheadleftarrow_{\mathcal{R}^\sharp{}_2} t$ by distinguishing cases according to the relative positions of o_1 and o_2 in s. The case where $o_1 = o_2$ is impossible as in the proof of Lemma 6.4, and for the cases where o_1 and o_2 are disjoint the proofs are trivial. We prove the remaining cases:

$o_1 < o_2$: Suppose $s \to t$ was an application of a (possibly nonlinear) $\mathcal{R}^\sharp{}_1$-rule $l \to r$ and $o_2 = o_1.p.q$, where $p \in O^{\mathcal{V}}(l)$. If l/p is a variable that occurs only once in l, then nothing is the matter; we reason exactly as in the proof of Lemma 6.4. If, however, the variable l/p occurs more than once in l then the reduction $s \xrightarrow{o_2}_{\mathcal{R}^\sharp{}_2} s'$ disturbs the redex s/o_1. Now observe that the redex can be restored by applying the same $\mathcal{R}^\sharp{}_2$-rule to other occurrences of s/o_2, so there is a term $s'' \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $s' \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} s''$ and s''/o_1 is again an l-redex. Contraction of this redex yields a term u that is reachable from t by contracting all descendants of the disjoint redexes contracted in the reduction $s \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} s''$.[4]

$o_2 < o_1$**:** Since $\mathfrak{R}$ has linear constructor variants, all rules in $\mathcal{R}^\sharp{}_2$ are left-linear. Therefore, in t/o_2 is still an $\mathcal{R}^\sharp{}_2$-redex. For the remainder of this case the proof is trivial.

Now the lemma follows by induction w.r.t. the length of the reduction $s \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} s'$. The case where $s \twoheadrightarrow^0_{\mathcal{R}^\sharp{}_2} s'$ is trivial. If $s \rightarrow^n_{\mathcal{R}^\sharp{}_2} u \rightarrow_{\mathcal{R}^\sharp{}_2} s'$ and the lemma holds for reductions of length n, then the induction hypothesis yields terms $u', v \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $u \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} u' \rightarrow_{\mathcal{R}^\sharp{}_1} v$ and $t \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} v$. By Proposition 7.9 there exists $v' \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $s' \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} v'$ and $u' \rightarrow^i_{\mathcal{R}^\sharp{}_2} v'$ with $i \in \{0,1\}$. If $i = 0$ then $u' = v'$, so we are done. If $i = 1$ then by the above there are $v'', w \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $v' \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} v'' \rightarrow_{\mathcal{R}^\sharp{}_1} w$ and $v \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} w$. The proof for this latter case is depicted in the diagram below.

$$\begin{tikzcd}
s \arrow[rrr, "\mathcal{R}^\sharp{}_1"] \arrow[d, two heads, "\mathcal{R}^\sharp{}_2"'] \arrow[drrr, phantom, "\text{(IH)}"] & & & t \arrow[d, two heads, "\mathcal{R}^\sharp{}_2"] \\
u \arrow[r, two heads, "\mathcal{R}^\sharp{}_2"] \arrow[d, "\mathcal{R}^\sharp{}_2"'] \arrow[dr, phantom, "\text{Prop. 7.9}"] & u' \arrow[rr, "\mathcal{R}^\sharp{}_1"] \arrow[d, "\mathcal{R}^\sharp{}_2"] & & v \arrow[d, two heads, "\mathcal{R}^\sharp{}_2"] \\
s' \arrow[r, two heads, "\mathcal{R}^\sharp{}_2"'] & v' \arrow[r, two heads, "\mathcal{R}^\sharp{}_2"'] & v'' \arrow[r, "\mathcal{R}^\sharp{}_1"'] & w
\end{tikzcd}$$

□

THEOREM 7.11 If $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ is a nonoverlapping TRS with linear constructor variants, then $\phi_{\mathfrak{R}}$ is an ω-simulation of $\mathfrak{R}$ by $\mathfrak{R}^\sharp$.

PROOF We need to prove that the two conditions of Definition 3.6 hold.
For the first condition we argue as in the proof of Theorem 6.5, using Lemma 7.10 instead of Lemma 6.4.
For the second condition we can apply the same argument as in the proof of Theorem 6.5, literally. □

It is an immediate consequence of the above theorem that Thatte's transformation preserves confluence for systems of the third type (Theorem 4.6). Moreover, it follows from the above theorem, using Theorem 6.6, that $\mathfrak{R}^\sharp$ is a correct transformation of $\mathfrak{R}$ if $\mathfrak{R}$ is a nonoverlapping TRS with linear constructor variants.

We now proceed with a proof that Thatte's transformation preserves the uniquely normalizing property for the class of weakly persistent systems. For the first time we will really need weak persistence. More specifically, we need the following consequence of weak persistence:

PROPOSITION 7.12 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a weakly persistent TRS.
If $s \in T_{\Sigma^\sharp,\mathcal{V}}$ is reachable from $T_{\Sigma,\mathcal{V}}$ and there is a $t \in T_{\Sigma,\mathcal{V}}$ such that $s^h \xrightarrow{r}_{\mathcal{R}} t$, then $s(\lambda) = s^h(\lambda)$.

PROOF Assume $s \in T_{\Sigma^\sharp,\mathcal{V}}$ reachable, $s = c_f(s_1, \ldots, s_n)$, $s^h = f(s^h_1, \ldots, s^h_n)$ (c_f the constructor variant of f) and $s^h \xrightarrow{r}_{\mathcal{R}} t$ for some $t \in T_{\Sigma,\mathcal{V}}$.
Since s is reachable, there is a $u \in T_{\Sigma,\mathcal{V}}$ such that $u \twoheadrightarrow_{\mathcal{R}^\sharp} s$ and at some point in this reduction an $\mathcal{R}^\sharp{}_2$-rule $l \rightarrow r$ must be applied to obtain the c_f-symbol. Note that without loss of generality we may assume that this application is at the root, so there are $u_1, \ldots, u_n \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $u \twoheadrightarrow_{\mathcal{R}^\sharp} l(u_1, \ldots, u_n) \rightarrow_{\mathcal{R}^\sharp{}_2} r(u_1, \ldots, u_n) \twoheadrightarrow_{\mathcal{R}^\sharp} s$.

[4]Note that, because of the way we defined this concept, the sets of descendants of these redexes over the reduction $s \rightarrow_{\mathcal{R}^\sharp{}_1} t$ are equal.

By construction of $\mathcal{R}^\sharp$, r consists entirely of constructor symbols and variables, so $r(u_1,\ldots,u_n) \xrightarrow[r]{nt}{}^*_{\mathcal{R}^\sharp} s$.
It is easy to see that then also $(r(u_1,\ldots,u_n))^h \xrightarrow[h(r)]{nt}{}^*_{\mathcal{R}} s^h$ (see Lemma 6.2(i); Thatte's proof of this lemma is applicable here). Moreover, by construction of $\mathcal{R}^\sharp$, r^h is a subtemplate of $\mathcal{R}$. However, then we have a contradiction with the assumption that $\mathfrak{R}$ is weakly persistent, because $s^h \xrightarrow{r}_{\mathcal{R}} t$. □

Definition 7.13 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a TRS.
We call $s \in T_{\Sigma^\sharp,\mathcal{V}}$ *balanced*, if $(u')^h = (u'')^h \Longrightarrow u' = u''$ for all subterms u', u'' of s. ⊣

Proposition 7.14 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a weakly persistent TRS and let $s \in T_{\Sigma^\sharp,\mathcal{V}}$ be a balanced term that is reachable from $T_{\Sigma,\mathcal{V}}$.
If $s^h \to_{\mathcal{R}} t$ for some $t \in T_{\Sigma,\mathcal{V}}$ by application of a rule $l \to r$ at occurrence $o \in O(s^h)$, then there is a substitution σ and an $l' \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $s/o = (l')^\sigma$, $(l')^h = l$, and $s \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} s[o \leftarrow (l^{ic})^\sigma] \to_{\mathcal{R}^\sharp{}_1} s[o \leftarrow r^\sigma]$.

Proof Let τ be such that $s^h/o = l^\tau$. Note that if $s^h/o_1 = s^h/o_2$, then $s/o_1 = s/o_2$ for all $o_1, o_2 \in O(s)$, since s is balanced, so there is an $l' \in T_{\Sigma^\sharp,\mathcal{V}}$ and a substitution σ such that $(l')^h = l$ and $s/o = (l')^\sigma$ (in particular, if l contains some variable x more than once, then l' contains x exactly at the same occurrences). Moreover, $l'(\lambda) = s(o) = s^h(o) = l^{ic}(o)$ by Proposition 7.12 and the construction of $\mathcal{R}^\sharp{}_1$. Therefore, $l' \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} l^{ic}$ by construction of $\mathcal{R}^\sharp{}_2$, so $s \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} s[o \leftarrow (l^{ic})^\sigma]$. Now $l^{ic} \to r$ is a rule in $\mathcal{R}^\sharp{}_1$, so $s[o \leftarrow (l^{ic})^\sigma] \to_{\mathcal{R}^\sharp{}_1} s[o \leftarrow r^\sigma]$. □

The following proposition is an immediate consequence of the above.

Proposition 7.15 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a weakly persistent TRS.
If $s \in \mathsf{NF}(\mathfrak{R}^\sharp)$ is reachable from $T_{\Sigma,\mathcal{V}}$ and balanced, then $s^h \in \mathsf{NF}(\mathfrak{R})$.

In the proof of the following lemma we will speak of 'linearized versions' of terms. We will informally introduce this notion. Let s be a term, possibly containing some variables more than once. s' is a *linearized version* of s if s' is a linear term that can be obtained from s by renaming occurrences of variables such that $o_1 \neq o_2 \Longrightarrow s/o_1 \neq s/o_2$ for all $o_1, o_2 \in O^{\mathcal{V}}(s)$. For instance, if $s = f(x,x,y,y)$ then $f(x_1,x_2,y_1,y_2)$ would be a linearized version of s. Of course, every term has infinitely many distinct linear versions. This will not impose a problem, because we are only interested in the instances of such linearized versions and all linearized versions of a term are instances of each other.

Lemma 7.16 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a weakly persistent and uniquely normalizing TRS.
For all $s, t \in \mathsf{NF}(\mathfrak{R}^\sharp)$ reachable from $T_{\Sigma,\mathcal{V}}$: $s^h = t^h \Longrightarrow s = t$.

Proof Fix $s, t \in \mathsf{NF}(\mathfrak{R}^\sharp[T_{\Sigma,\mathcal{V}}])$ and assume that $s^h = t^h$, but $s \neq t$; we derive a contradiction.
Since both s and t are in normal form, all their subterms are in normal form. Therefore, without loss of generality, we may assume that s and t are balanced terms (if s is not balanced, then there are subterms u', u'' of s such that $u' \neq u''$, while $(u')^h = (u'')^h$, so we take $s = u'$ and $t = u''$; if t is not balanced then we apply a similar argument; repeating this will eventually yield balanced terms for s and t, because we only consider finite terms).
Moreover, because trivially all subterms of a balanced term are balanced, we may assume without loss of generality that s and t are minimal in the sense that s and t only differ at the root, i.e. $o \neq \lambda \Longrightarrow s/o = t/o$ for all $o \in O(s), O(t)$ (again we can find such minimal terms by inspection of s and t: if s, t are not minimal, then for some $o \neq \lambda$ we have $s/o \neq t/o$, so we take $s = s/o$ and $t = t/o$; this will eventually yield that s, t are minimal, by finiteness of terms).

If s and t are minimal, then $s = c_f(s_1, \ldots, s_n)$ and $t = f(t_1, \ldots, t_n)$ and $s_i = t_i$ for all $1 \leq i \leq n$ (or vice versa, but in that case we just interchange s and t).
Since s is reachable, there is a $u \in T_{\Sigma,\mathcal{V}}$ such that $u \twoheadrightarrow_{\mathcal{R}^\sharp} s$ and at some point in this reduction some $\mathcal{R}^\sharp{}_2$-rule $l \rightarrow r$ is applied to obtain the c_f-symbol. Without loss of generality we may assume that this application is at the root, so there are $u_1, \ldots, u_n \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $u \twoheadrightarrow_{\mathcal{R}^\sharp} l(u_1, \ldots, u_m) \rightarrow_{\mathcal{R}^\sharp{}_2} r(u_1, \ldots, u_m) \twoheadrightarrow_{\mathcal{R}^\sharp} s$.
Moreover, by construction of $\mathcal{R}^\sharp{}_2$, all symbols at nonvariable positions in r are constructor symbols, so $r(u_1, \ldots, u_m) \xrightarrow[r]{nt}{}^*_{\mathcal{R}^\sharp} s$. Therefore, s is an instance of a linearized version of r and t is an instance of a linearized version of l (using minimality and the construction of $\mathcal{R}^\sharp{}_2$). Since by assumption t was in normal form, it cannot be an instance of l, so l contains at least one variable more than once, say at occurrences $o_1, o_2 \in O^{\mathcal{V}}(l)$, and $t/o_1 \neq t/o_2$. This implies $s/o_1 \neq s/o_2$.
However, $l(u_1, \ldots, u_m)/o_1 = l(u_1, \ldots, u_m)/o_2 = u_i$ for some $1 \leq i \leq m$, so, because $l(u_1, \ldots, u_m) \rightarrow_{\mathcal{R}^\sharp{}_2} r(u_1, \ldots, u_m) \xrightarrow[r]{nt}{}^*_{\mathcal{R}^\sharp} s$, we find $s/o_2 \twoheadleftarrow_{\mathcal{R}^\sharp} u_i \twoheadrightarrow_{\mathcal{R}^\sharp} s/o_1$. From this $(s/o_2)^h \twoheadleftarrow_{\mathcal{R}} u_i^h \twoheadrightarrow_{\mathcal{R}} (s/o_1)^h$ follows (using Lemma 6.2(i)).
Since s is balanced $(s/o_1)^h \neq (s/o_2)^h$. Moreover, since $s/o_1, s/o_2 \in \mathsf{NF}(\mathfrak{R}^\sharp[T_{\Sigma,\mathcal{V}}])$ are balanced terms, we find $(s/o_1)^h, (s/o_2)^h \in \mathsf{NF}(\mathfrak{R})$ (using Proposition 7.15). Now we have reached a contradiction, because by assumption $\mathfrak{R}$ was uniquely normalizing. □

Corollary 7.17 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a weakly persistent and uniquely normalizing TRS. If $s \in \mathsf{NF}(\mathfrak{R}^\sharp)$ is reachable from $T_{\Sigma,\mathcal{V}}$, then s is balanced.

Proof Contraposition. If s is not balanced, then there are subterms u', u'' of s such that $(u')^h = (u'')^h$ and $u' \neq u''$, then u' and u'' cannot be both in normal form, for this would contradict Lemma 7.16. However, in that case s is not in normal form either. □

Corollary 7.18 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a weakly persistent, uniquely normalizing TRS. If $n \in \mathsf{NF}(\mathfrak{R}^\sharp)$, $n' \in T_{\Sigma^\sharp,\mathcal{V}}$ are both reachable from $T_{\Sigma,\mathcal{V}}$ and $n^h = (n')^h$, then $n' \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} n$.

Proof Since $s \rightarrow_{\mathcal{R}^\sharp{}_1} t \Longrightarrow s^h \rightarrow_{\mathcal{R}} t^h$, and $\mathfrak{R}^\sharp_2$ is strongly normalizing: $\mathsf{nf}_{\mathcal{R}^\sharp{}_2}(n') \in \mathsf{NF}(\mathfrak{R}^\sharp)$. Clearly, $(\mathsf{nf}_{\mathcal{R}^\sharp{}_2}(n'))^h = n^h$, so by Lemma 7.16 $\mathsf{nf}_{\mathcal{R}^\sharp{}_2}(n') = n$. □

Theorem 7.19 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a weakly persistent TRS; the simulations $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h \rangle$ and $\phi_{\mathfrak{R}}$ preserve the unique normal form property.

Proof First we prove that $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h \rangle$ preserves UN.
Fix $s \in T_{\Sigma,\mathcal{V}}$ and suppose $\mathfrak{R} \models \mathsf{UN}(s)$. In order to prove that $\mathfrak{R}^\sharp \models \mathsf{UN}(s)$ we show that if $n_1 \twoheadleftarrow_{\mathcal{R}^\sharp} s \twoheadrightarrow_{\mathcal{R}^\sharp} n_2$, then $n_1 = n_2$.
Note that by Corollary 7.17 n_1 and n_2 are both balanced. Using Lemma 6.2(i), we find that $s \twoheadrightarrow_{\mathcal{R}} n_1^h$ and $s \twoheadrightarrow_{\mathcal{R}} n_2^h$. Moreover, $n_1^h, n_2^h \in \mathsf{NF}(\mathfrak{R})$ by Proposition 7.15. Since $\mathfrak{R}$ is uniquely normalizing, $n_1^h = n_2^h$ and thus, by Lemma 7.16, $n_1 = n_2$, so $\mathfrak{R}^\sharp \models \mathsf{UN}(s)$.
So $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h \rangle$ preserves UN, and since $s \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} \phi_{\mathfrak{R}}(s)$, $\phi_{\mathfrak{R}}$ preserves UN as well. □

Lemma 7.20 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a weakly persistent, uniquely normalizing TRS. If $\mathfrak{R} \models \mathsf{WN}$, then $\mathfrak{R}^\sharp[T_{\Sigma,\mathcal{V}}] \models \mathsf{WN}$.

Proof We inductively define for $s \in T_{\Sigma^\sharp,\mathcal{V}}$:

1. if $s \in \Sigma \cup \mathcal{V}$, then $\mathsf{depth}(s) = 0$; and
2. if $s = f(s_1, \ldots, s_j)$ and $f \in \Sigma^\sharp$, then $\mathsf{depth}(s) = 1 + \max_{1 \leq i \leq j}(\mathsf{depth}(s_i))$.

Now we show by induction to $\mathsf{depth}(s)$ that $\mathfrak{R}^\sharp \models \mathsf{WN}(s)$ for all s reachable from $T_{\Sigma,\mathcal{V}}$. If $\mathsf{depth}(s) = 0$, then $s^h = s$ (recall that $\Sigma^\sharp - \Sigma$ does not contain constant symbols). Because $\mathfrak{R} \models \mathsf{WN}(s)$, $s \twoheadrightarrow_{\mathcal{R}} \mathsf{nf}_{\mathfrak{R}}(s)$, so $s \twoheadrightarrow_{\mathcal{R}^\sharp} \mathsf{nf}_{\mathfrak{R}}(s)$ by Lemma 6.2(ii) and $\mathsf{nf}_{\mathfrak{R}}(s) \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} \mathsf{nf}_{\mathfrak{R}^\sharp}(s)$ by completeness of $\mathfrak{R}^\sharp_2$.

Suppose that $\mathfrak{R}^\sharp \models \mathsf{WN}(s')$ for all $s' \in T_{\Sigma^\sharp,\mathcal{V}}$ with $\mathsf{depth}(s') \leq k_1$ (IH1). Let $s = f(s_1,\ldots,s_j)$ with $\mathsf{depth}(s) = k_1 + 1$. By induction hypothesis there are $n_1,\ldots,n_j \in \mathsf{NF}(\mathfrak{R}^\sharp)$ such that $s \twoheadrightarrow_{\mathcal{R}^\sharp} f(n_1,\ldots,n_j)$ for $1 \leq i \leq j$. We distinguish two cases: (1) $f \in \Sigma^\sharp - \Sigma$; and (2) $f \in \Sigma$. The first case is trivial since $f(n_1,\ldots,n_m)$ is already in $\mathfrak{R}^\sharp$-normal form. Therefore it remains to show that $\mathfrak{R}^\sharp \models \mathsf{WN}(f(n_1,\ldots,n_j))$ for $f \in \Sigma$.
Instead of proving $\mathfrak{R}^\sharp \models \mathsf{WN}(f(n_1,\ldots,n_j))$ directly, we prove $\mathfrak{R}^\sharp \models \mathsf{WN}(t(n_1,\ldots,n_m))$ for all $t \in T_{\Sigma,\mathcal{V}}$, $n_1,\ldots,n_m \in \mathsf{NF}(\mathfrak{R}^\sharp)$ with $\mathsf{depth}(n_i) \leq k_1$ $(1 \leq i \leq m)$. Obviously, $\mathfrak{R}^\sharp \models \mathsf{WN}(f(n_1,\ldots,n_j))$ then follows.
Induction to the length of the reduction $(t(n_1,\ldots,n_m))^h \twoheadrightarrow_{\mathcal{R}} \mathsf{nf}_{\mathfrak{R}}((t(n_1,\ldots,n_m))^h)$.
If $(t(n_1,\ldots,n_m))^h \in \mathsf{NF}(\mathfrak{R})$, then $t(n_1,\ldots,n_m) \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} \mathsf{nf}_{\mathfrak{R}^\sharp}(t(n_1,\ldots,n_m))$ follows by completeness of $\mathfrak{R}^\sharp_2$.
Suppose for reductions of length $\leq k_2$ it holds (IH2) and assume $(t(n_1,\ldots,n_m))^h \rightarrow_{\mathcal{R}} t' \rightarrow^{k_2}_{\mathcal{R}} \mathsf{nf}_{\mathfrak{R}}((t(n_1,\ldots,n_m))^h)$ with $t' \in T_{\Sigma,\mathcal{V}}$.
Observe that if $t(n_1,\ldots,n_m)$ is not balanced, then it contains subterms u_1, u_2 such that $u_1^h = u_2^h$, while $u_1 \neq u_2$. Note that $O(u_1) = O(u_2)$ and let $o \in O(u_1)$ be a minimal position such that $u_1(o) \neq u_2(o)$, i.e. there is no $q \in O(u_1)$ such that $u_1(o) \neq u_2(o)$ and $q < o$. Now, $u_1/o = f(v_1,\ldots,v_l)$ and $u_2/o = c_f(w_1,\ldots,w_l)$ for some $f \in \Sigma$ with constructor variant c_f (it may be necessary to interchange u_1 and u_2 to arrive in this situation). Moreover $u_2/o \in \mathsf{NF}(\mathfrak{R}^\sharp)$, since t does not contain symbols of the form c_f. Therefore, $u_1/o \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} u_2/o$ by Corollary 7.18. Clearly, there are finitely many such o and they are all disjoint, so by a repetition of this argument we obtain that there is a balanced $t'' \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $t(n_1,\ldots,n_m) \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t''$. Notice that this t'' does not contain subterms u with $u(\lambda) \in \Sigma^\sharp - \Sigma$ and $\mathsf{depth}(u) > k_1$.
Since t'' is balanced, by Proposition 7.14 there are $t''', t^* \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $t'' \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t''' \rightarrow_{\mathcal{R}^\sharp{}_1} t^*$ and $(t^*)^h = t'$. It is easy to verify that t^* still does not contain subterms u with $u(\lambda) \in \Sigma^\sharp - \Sigma$ and $\mathsf{depth}(u) > k_1$. Therefore, (IH2) is applicable and yields $\mathfrak{R}^\sharp \models \mathsf{WN}(t^*)$ and since $t(n_1,\ldots,n_m) \twoheadrightarrow_{\mathcal{R}^\sharp} t^*$ we find $\mathfrak{R}^\sharp \models \mathsf{WN}(t(n_1,\ldots,n_m))$. □

Theorem 7.21 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a weakly persistent, weakly normalizing TRS; the simulations $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h \rangle$ and $\phi_{\mathfrak{R}}$ preserve confluence.

Proof If $\mathfrak{R} \models \mathsf{CR}$ and $\mathfrak{R} \models \mathsf{WN}$, then by Theorem 7.19 $\mathfrak{R}^\sharp \models \mathsf{UN}$ and by Lemma 7.20 $\mathfrak{R}^\sharp \models \mathsf{WN}$. Since $\mathsf{WN} \,\&\, \mathsf{UN} \implies \mathsf{CR}$, we find $\mathfrak{R}^\sharp \models \mathsf{CR}$ for terms reachable from $T_{\Sigma,\mathcal{V}}$. Thus, both simulations preserve confluence. □

Corollary 7.22 Let $\mathfrak{R}$ be a weakly persistent TRS; the simulations $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h \rangle$ and $\phi_{\mathfrak{R}}$ preserve semi-completeness.

Trivially, if $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a weakly persistent, strongly normalizing TRS,the the simulations $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h \rangle$ and $\phi_{\mathfrak{R}}$ preserve confluence. Moreover, by Corollary 7.6, systems that are nonoverlapping, strongly normalizing and have confluent root overlaps are weakly persistent. Therefore it now follows that Thatte's transformation preserves confluence for systems of the second type.

We conclude this section with a proof that Thatte's transformation is correct according to [FP96] for weakly persistent, uniquely normalizing TRSs.

Theorem 7.23 (Correctness) If $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ is a weakly persistent and uniquely normalizing TRS, then $\mathfrak{R}^\sharp$ is a correct transformation of $\mathfrak{R}$ as in [FP96].

Proof By Theorem 6.6 the simulation $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h \rangle$ preserves termination, so it suffices to show that it also preserves the normal form relation (Definition 3.2). Note that h is defined for all terms in $T_{\Sigma^\sharp,\mathcal{V}}$, so trivially it is defined on all reachable normal forms in $\mathfrak{R}^\sharp$.
$\{n^h \mid n \in \mathsf{nf}_{\mathfrak{R}^\sharp}[s]\} \subseteq \mathsf{nf}_{\mathfrak{R}}[s]$: If $s \twoheadrightarrow_{\mathcal{R}^\sharp} n$ and $n \in \mathsf{NF}(\mathfrak{R}^\sharp)$, then by Corollary 7.17 n is a

balanced term, so by Proposition 7.15 $n^h \in \mathsf{NF}(\mathfrak{R})$. Moreover, by Lemma 6.2(i) $s \twoheadrightarrow_{\mathcal{R}} n^h$, so $n^h \in \mathsf{nf}_{\mathfrak{R}}[s]$.
$\mathsf{nf}_{\mathfrak{R}}[s] \subseteq \{n^h \mid n \in \mathsf{nf}_{\mathfrak{R}^\sharp}[s]\}$: If $s \twoheadrightarrow_{\mathcal{R}} n'$ and $n' \in \mathsf{NF}(\mathfrak{R})$, then by Lemma 6.2(ii) $s \twoheadrightarrow_{\mathcal{R}^\sharp} n'$ and using Lemma 6.2(i) and completeness of $\mathfrak{R}^\sharp_2$ we find $n' \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} n$ with $n \in \mathsf{nf}_{\mathfrak{R}^\sharp}[s]$ and clearly $n' = n^h$. □

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A Counterexamples for Verma's Proof

In this section we present the counterexamples we found for a remark in Verma's proof of [Ver95b, Lemma 13] and his [Ver95b, Lemma 12].

In the proof of his Lemma 13 [Ver95b] Verma claimes that if $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ is a weakly persistent system with transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, {\mathcal{R}^\sharp}_1 \cup {\mathcal{R}^\sharp}_2 \rangle$ and $s \in T_{\Sigma^\sharp,\mathcal{V}}[T_{\Sigma,\mathcal{V}}]$, then $s^h \twoheadrightarrow_{{\mathcal{R}^\sharp}_2} s$ in $\mathfrak{R}^\sharp$. The following example demonstrates that this is not true.

Example A.1 Consider the TRS $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$, with transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp \rangle$. Here $\Sigma = \{a, b, f, g, h\}$ and $\Sigma^\sharp = \Sigma \cup \{c_f\}$. $\mathcal{R}$ and $\mathcal{R}^\sharp$ respectively consist of the rules:

$$\mathcal{R} = \left\{ \begin{array}{lll} g(f(x,x)) & \to & a, \\ f(a,b) & \to & a, \\ h(x) & \to & b \end{array} \right\}, \quad \mathcal{R}^\sharp = \left\{ \begin{array}{lll} g(c_f(x,x)) & \to & a, \\ f(a,b) & \to & a, \\ h(x) & \to & b, \\ f(x,x) & \to & c_f(x,x) \end{array} \right\}$$

Observe that $\mathfrak{R} \models \mathsf{SN}$[5] and $\mathfrak{R}$ is nonambiguous. Applying Corollary 5.3 and Newman's Lemma we find $\mathfrak{R} \models \mathsf{CR}$, so $\mathfrak{R}$ has the unique normal form property. Now, if $\mathfrak{R}$ would not be weakly persistent, then there would be some term $t \in T_{\Sigma,\mathcal{V}}$ such that $f(t,t) \twoheadrightarrow_{\mathcal{R}} f(a,b)$. However, a and b are both normal forms in $\mathfrak{R}$, so that $a \twoheadleftarrow_{\mathcal{R}} t \twoheadrightarrow_{\mathcal{R}} b$, which contradicts $\mathfrak{R} \models \mathsf{UN}$. Therefore, $\mathfrak{R}$ is weakly persistent.

In $\mathfrak{R}^\sharp$ we have $f(h(b),h(b)) \to_{{\mathcal{R}^\sharp}_2} c_f(h(b),h(b)) \to_{{\mathcal{R}^\sharp}_1} c_f(h(b),b)$. This reduction demonstrates that the term $c_f(h(b),b)$ is reachable, under both simulation mappings $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h \rangle$ and ϕ (for the latter note that $\phi(f(h(b),h(b))) = c_f(h(b),h(b))$). But $(c_f(h(b),b))^h = f(h(b),b)$ is in ${\mathcal{R}^\sharp}_2$-normal form.

In fact, the above example also demonstrates that weak persistence is not sufficient to establish $\mathfrak{R} \rightleftarrows^{\phi} \mathfrak{R}^\sharp$. In $\mathfrak{R}$ we have $f(h(b),h(b)) \to_{\mathcal{R}} f(h(b),b)$, but $\phi(f(h(b),b)) = f(h(b),b)$ and in $\mathfrak{R}^\sharp$ we do not have $c_f(h(b),h(b)) \to^{+}_{\mathcal{R}^\sharp} f(h(b),b)$. This is a violation of the first condition of Definition 3.6. Recall that $\mathfrak{R} \rightleftarrows^{\phi} \mathfrak{R}^\sharp$ would have implied confluence right away (Theorem 4.6).

We now adapt the above counterexample a little to obtain a counterexample to Verma's Lemma 12:

Claim A.1 (Lemma 12 in [Ver95b]) Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a weakly persistent TRS with transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp \rangle$.

If $s \in T_{\Sigma^\sharp,\mathcal{V}}[T_{\Sigma,\mathcal{V}}], t \in T_{\Sigma,\mathcal{V}}$ and $s^h \to_{\mathcal{R}} t$, then there is a $t' \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $s \twoheadrightarrow_{\mathcal{R}^\sharp} t'$ and $(t')^h = t$.

Example A.2 Consider the TRS $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$, with transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp \rangle$. Here $\Sigma = \{a, b, f, g, h\}$ and $\Sigma^\sharp = \Sigma \cup \{c_f\}$. $\mathcal{R}$ and $\mathcal{R}^\sharp$ respectively consist of the rules:

$$\mathcal{R} = \left\{ \begin{array}{lll} g(f(x,x)) & \to & a, \\ f(a,b) & \to & a, \\ h(x,x) & \to & x \end{array} \right\}, \quad \mathcal{R}^\sharp = \left\{ \begin{array}{lll} g(c_f(x,x)) & \to & a, \\ f(a,b) & \to & a, \\ h(x,x) & \to & x, \\ f(x,x) & \to & c_f(x,x) \end{array} \right\}$$

It is easy to adapt the argument of Example A.1 to obtain $\mathfrak{R} \models \mathsf{SN}$, $\mathfrak{R} \models \mathsf{CR}$, $\mathfrak{R} \models \mathsf{UN}$ and weak persistence.

The term $c_f(h(b,b),b)$ is reachable, as witnessed by the reduction:

$$f(h(b,b),h(b,b)) \to_{{\mathcal{R}^\sharp}_2} c_f(h(b,b),h(b,b)) \to_{{\mathcal{R}^\sharp}_1} c_f(h(b,b),b)$$

[5] If we take $||s|| = |O^{\mathcal{F}}(s)|$ as a complexity measure, then $s \to_{\mathcal{R}} t$ in $\mathcal{R}$ implies $|t| < |s|$.

Furthermore, in $\mathfrak{R}$ we have $h(f(h(b,b),b), f(h(b,b),b)) \to_{\mathcal{R}} f(h(b,b),b)$. Now, take $s = h(f(h(b,b),b), c_f(h(b,b),b))$; it is clear that $s^h = h(f(h(b,b),b), f(h(b,b),b))$, so Verma's Lemma 12 suggests the existence of a $t' \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $s \twoheadrightarrow_{\mathcal{R}^\sharp} t'$ and $(t')^h = f(h(b,b),b)$.

However, there is no such t'. We argue as follows. First note that s is not a redex of $\mathcal{R}^\sharp$ (while s^h is a redex of $\mathcal{R}$). The only redexes in s are the $h(b,b)$ at occurrences 1.1 and 2.1. Moreover, contraction of either one of these two redexes does not yield any new redexes and neither of the resulting terms (say s') satisfy $(s')^h = f(h(b,b),b)$. So in our search for such a term we will have to contract them both. That is, we get the reduction $h(f(h(b,b),b), c_f(h(b,b),b)) \twoheadrightarrow_{\mathcal{R}^\sharp{}_1} h(f(b,b), c_f(b,b))$. Still, we have not found our t', because $h(f(b,b), c_f(b,b))^h \neq f(h(b,b),b)$, but we created one new redex: $f(b,b)$. Thus, the only possibility we have is to contract this redex: $h(f(b,b), c_f(b,b)) \to_{\mathcal{R}^\sharp{}_2} h(c_f(b,b), c_f(b,b))$. Again, this reduction does not yield the t' we were looking for, so we contract this whole term (note that this is the only redex): $h(c_f(b,b), c_f(b,b)) \to_{\mathcal{R}^\sharp{}_1} c_f(b,b)$. Since, $c_f(b,b)$ is a normal form, it is our last option for t', but $c_f(b,b)^h = f(b,b) \neq f(h(b,b),b)$. This refutes Verma's claim.

B A Confluence Proof for Verma's Counterexample

Lemma B.1 The TRS $\mathfrak{R} = \langle \Sigma, \mathcal{R}^c \cup \mathcal{R}^f \rangle$, where $\Sigma = \{a, c, f, g, h\}$, $\mathcal{R}^c = \{c \to g(c)\}$ and $\mathcal{R}^f = \{h(f(x,x)) \to h(a),\ f(x, g(x)) \to a\}$ is confluent.

Proof Note that if $s \to_{\mathcal{R}^c} t$, then $O^{\mathcal{F}}(s) \subset O^{\mathcal{F}}(t)$. Now, we will first prove three simple facts:

1. $\langle \Sigma, \mathcal{R}^c \rangle \models \mathsf{CR}$.

   Because it is clearly regular, this is an immediate consequence of Corollary 5.4.

2. $\forall s, t_1, t_2 \in T_{\Sigma,\mathcal{V}} (t_2 \twoheadleftarrow_{\mathcal{R}^c} s \xrightarrow{o}_{\mathcal{R}^f} t_1 \Longrightarrow \exists t, t' \in T_{\Sigma,\mathcal{V}} (t_2 \twoheadrightarrow_{\mathcal{R}^c} t' \xrightarrow{o}_{\mathcal{R}^f} t \twoheadleftarrow_{\mathcal{R}^c} t_1)$, and for all positions $o_1, \ldots, o_n$ of redexes contracted in the reduction $t_2 \twoheadrightarrow_{\mathcal{R}^c} t'$, it holds that $o_i > o$ $(1 \leq i \leq n)$.

   Either $s/o = f(u, g(u))$ or $s/o = h(f(u,u))$. Note that $o \in O^{\mathcal{F}}(t_2)$ and that $t_2/o = f(u_1, g(u_2))$ or $t_2/o = h(f(u_1, u_2))$. If $u_1 = u_2$, then the same rule as applied in $s \to_{\mathcal{R}^f} t_1$ is still applicable. In this case we take $t' = t_2$. If $u_1 \neq u_2$, then they have a common reduct u' by the previous fact. We take $t' = f(u', g(u'))$ or $t' = h(f(u', u'))$ respectively, and $t_2 \twoheadrightarrow_{\mathcal{R}^c} t'$. The redexes contracted in this reduction clearly all lie below o, and the $\mathcal{R}^f$-rule can be applied to t'/o, so $t' \to_{\mathcal{R}^f} t$. Finally, contracting all descendants over $s \to_{\mathcal{R}^f} t_1$ of redexes contracted in the reduction $s \twoheadrightarrow_{\mathcal{R}^c} t_2 \twoheadrightarrow_{\mathcal{R}^c} t'$, results in $t_1 \twoheadrightarrow_{\mathcal{R}^c} t$.

3. For all $s, t_1, t_1', t_2, t_2' \in T_{\Sigma,\mathcal{V}}$: $t_1 \xleftarrow{o_1}_{\mathcal{R}^f} t_1' \twoheadleftarrow_{\mathcal{R}^c} s \twoheadrightarrow_{\mathcal{R}^c} t_2' \xrightarrow{o_2}_{\mathcal{R}^f} t_2$, with for all redex positions $p_1, \ldots, p_m$ and $q_1, \ldots, q_n$ contracted in $s \twoheadrightarrow_{\mathcal{R}^c} t_1'$ and $s \twoheadrightarrow_{\mathcal{R}^c} t_2'$, respectively, $p_i > o_1$ and $q_j > o_2$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$, implies that there is a $t \in T_{\Sigma,\mathcal{V}}$, such that $t_1 \twoheadrightarrow_{\mathcal{R}} t \twoheadleftarrow_{\mathcal{R}} t_2$.

   The cases where $o_1 \not< o_2$ and $o_2 \not< o_1$ (so either $o_1 = o_2$, or they are disjunct) are trivial. Moreover, if $o_1 < o_2$, or $o_2 < o_1$, and $s/o_1 = h(s/o_2)$, or $s/o_2 = h(s/o_1)$, then we find that $t_1/o_1 = h(a) = t_2/o_1$, or $t_1/o_2 = h(a) = t_2/o_2$, respectively, and thus $t_1 = t_2$.

   So assume $o_1 < o_2$ and not $s/o_1 = h(s/o_2)$. Either $s/o_1 = f(u, g(u))$ or $s/o_1 = h(f(u,u))$, and $t_2/o_1 = f(u_1, g(u_2))$, or $t_2/o_1 = h(f(u_1, u_2))$, with $u = u_1$, or $u = u_2$. Clearly, we can reduce t_2/o_1 to $f(u', g(u'))$, or to $h(f(u', u'))$, using the same rules as applied in $s \twoheadrightarrow_{\mathcal{R}^c} t_2' \xrightarrow{o_2}_{\mathcal{R}^f} t_2$ (either $u' = u_1$, or $u' = u_2$) to obtain a term t', and then apply the same rule as applied in $t_1' \xrightarrow{o_1}_{\mathcal{R}^f} t_1$. This gives us the reduction $t_2 \twoheadrightarrow_{\mathcal{R}} t' \xrightarrow{o_1}_{\mathcal{R}^f} t_1$.

   The case where $o_2 < o_1$ and not $s/o_2 = h(s/o_1)$ goes similarly.

We define on $s \in T_{\Sigma,\mathcal{V}}$ a complexity measure $||s|| = |\{o \in O^{\mathcal{F}}(s) \mid s(o) = f\}|$. Clearly $||s|| < \omega$ for all $s \in T_{\Sigma,\mathcal{V}}$ and if $s \to_{\mathcal{R}^f} s'$, then $||s|| < ||s'||$. With the above facts at hand, we will derive confluence for all terms $\forall s \in T_{\Sigma,\mathcal{V}} (\mathfrak{R} \models \mathsf{CR}(s))$, using induction w.r.t. $||s||$.
The base case follows immediately from the first fact, for if $t_1 \twoheadleftarrow_{\mathcal{R}} s \twoheadrightarrow_{\mathcal{R}} t_2$ and $||s|| = 0$, then $s \twoheadrightarrow_{\mathcal{R}^c} t_1$ and $s \twoheadrightarrow_{\mathcal{R}^c} t_2$.
Assume that $t_1 \twoheadleftarrow_{\mathcal{R}} s \twoheadrightarrow_{\mathcal{R}} t_2$, with $||s|| = n+1$ and $\forall t \in T_{\Sigma,\mathcal{V}} (\mathfrak{R} \models \mathsf{CR}(t))$ with $||t|| = n$ (IH).
Note that if both reductions only contain applications of $\mathcal{R}^c$-rules, then we get a common reduct using the first fact. If only one of the reductions contains applications of $\mathcal{R}^f$-rules, then we have the following diagram:

$$\begin{tikzcd}
s \arrow[r, two heads, "\mathcal{R}^c"] \arrow[d, two heads, "\mathcal{R}^c"'] \arrow[dr, phantom, "(1)"] & \cdot \arrow[rr, "\mathcal{R}^f"] \arrow[d, two heads, "\mathcal{R}^c"] \arrow[drr, phantom, "(2)"] & & \cdot \arrow[r, two heads, "\mathcal{R}"] \arrow[d, two heads, "\mathcal{R}^c"'] \arrow[dr, phantom, "(IH)"] & t_1 \arrow[d, two heads, "\mathcal{R}"] \\
t_2 \arrow[r, two heads, "\mathcal{R}^c"'] & \cdot \arrow[r, two heads, "\mathcal{R}^c"'] & \cdot \arrow[r, "\mathcal{R}^f"'] & \cdot \arrow[r, two heads, "\mathcal{R}"'] & t
\end{tikzcd}$$

If both reductions contain applications of $\mathcal{R}^f$-rules then we can tile the following diagram:

$$\begin{tikzcd}
s \arrow[r, two heads, "\mathcal{R}^c"] \arrow[d, two heads, "\mathcal{R}^c"'] \arrow[dr, phantom, "(1)"] & \cdot \arrow[rr, "\mathcal{R}^f"] \arrow[d, two heads, "\mathcal{R}^c"] \arrow[drr, phantom, "(2)"] & & \cdot \arrow[r, two heads, "\mathcal{R}"] \arrow[d, two heads, "\mathcal{R}^c"'] \arrow[dddr, phantom, "(IH)"] & t_1 \arrow[ddd, two heads, "\mathcal{R}"] \\
\cdot \arrow[r, two heads, "\mathcal{R}^c"'] \arrow[dd, "\mathcal{R}^f"'] \arrow[ddr, phantom, "(2)"] & \cdot \arrow[r, two heads, "\mathcal{R}^c"'] \arrow[d, two heads, "\mathcal{R}^c"] \arrow[ddrr, phantom, "(3)"] & \cdot \arrow[r, "\mathcal{R}^f"'] & \cdot \arrow[dd, two heads, "\mathcal{R}"'] & \\
& \cdot \arrow[d, "\mathcal{R}^f"] & & & \\
\cdot \arrow[r, two heads, "\mathcal{R}^c"] \arrow[d, two heads, "\mathcal{R}"'] \arrow[drrrr, phantom, "(IH)"] & \cdot \arrow[rr, two heads, "\mathcal{R}"] & & \cdot \arrow[r, two heads, "\mathcal{R}"] & \cdot \arrow[d, two heads, "\mathcal{R}"] \\
t_2 \arrow[rrrr, two heads, "\mathcal{R}"'] & & & & t
\end{tikzcd}$$

□