TERMINATION AND CONFLUENCE IN INFINITARY TERM REWRITING

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Abstract. The basic notions of the theory of term rewriting are defined for terms that may involve function letters of infinite arity. A sufficient condition for completeness is derived, and its use demonstrated by the example of abstract clones over infinitary signatures.

Introduction

In general, to see that two terms of some equational theory $\mathcal{A}$ are not equivalent requires a model of $\mathcal{A}$ in which their interpretations are distinct. The construction of such models is often bothersome. It may be avoided when the axioms of $\mathcal{A}$ can be oriented in such a way that they form a semi-complete term rewriting system. The normal forms then constitute a canonical model of $\mathcal{A}$, which is sufficient for all demonstrations of nonequivalence (see [2]).

A theory of rewriting for infinite terms, that is, terms constructed with function symbols of infinite arity, does not have the straightforward calculational motivation of the well-known finitary term rewriting (cf. [2] and [4]). It would still have its use, however, in demonstrating nonequivalence. For example, the theory of categories with certain infinite products may be axiomatized by an infinite set of infinitary equations. (Cf. [5] for the finite case.) One way of showing terms of this theory to be nonequivalent would be to interpret them as distinct arrows in some category of sets. As an application of the results of this note, we shall derive another method: the theory in question may be so oriented as to constitute a complete term rewriting system, and we can decide equivalence, for sufficiently conceivable terms, by normalizing and comparing the normal forms.

The subject of the present note is essentially different from the infinitary term rewriting of Kennaway c.s. [3]. The infinite reduction sequences that they study will here be considered simply divergent.

Terms

By a *vocabulary* let us understand a set $\mathcal{F}$ of function letters, each associated with a fixed ordinal which we call its *arity*. If $\mathcal{F}$ is a vocabulary, then the set $T_{\mathcal{F}}$ of *terms over* $\mathcal{F}$ is the least set T such that for any α-ary $F \in \mathcal{F}$ and any α-

termed sequence $\langle t_\xi | \xi < \alpha \rangle$ of elements of T, $F(t_\xi | \xi < \alpha) \in T$. We picture $F(t_\xi | \xi < \alpha)$ as an oriented tree with a root labeled F from which the t_ξ grow, from left to right as ξ increases. *Variables* are special nullary function symbols. If X is a set of variables, then $T_{\mathcal{F},X}$, the set of *terms over $\mathcal{F}$ in variables from X*, is $T_{\mathcal{F} \cup X}$.

Let t be a term over $\mathcal{F}$ (in variables from X), and $\overline{\xi} = \langle \xi_i | i < n \rangle$ a finite sequence of ordinals. The *label* $\mathrm{Lab}(t, \overline{\xi})$ of t at $\overline{\xi}$, and the *subterm* $\mathrm{Sub}(t, \overline{\xi})$ of t at $\overline{\xi}$, are defined inductively as follows:
(i) $\mathrm{Sub}(t, \varnothing) = t$ ($\varnothing$ is the empty sequence), and if $t \in X$, $\mathrm{Lab}(t, \varnothing) = t$, while if $t = F(t_\xi | \xi < \alpha)$ with $F \in \mathcal{F}$, $\mathrm{Lab}(t, \varnothing) = F$;
(ii) $\mathrm{Lab}(F(t_\xi | \xi < \alpha), \langle \gamma, \delta_1, \ldots, \delta_n \rangle) = \mathrm{Lab}(t_\gamma, \langle \delta_1, \ldots, \delta_n \rangle)$ and
$\mathrm{Sub}(F(t_\xi | \xi < \alpha), \langle \gamma, \delta_1, \ldots, \delta_n \rangle) = \mathrm{Sub}(t_\gamma, \langle \delta_1, \ldots, \delta_n \rangle)$.

A sequence $\overline{\xi}$ for which $\mathrm{Lab}(t, \overline{\xi})$ is defined will be called a *position in t*. For any $\overline{\xi}$ and $F \in \mathcal{F}$ we define

$$U(F, \overline{\xi}) := \{ t \in T_{\mathcal{F}} | \ \mathrm{Lab}(t, \overline{\xi}) = F \}.$$

We topologize $T_{\mathcal{F}}$ by taking the $U(F, \overline{\xi})$ as subbasic opens. Then t is the limit of a sequence $\langle t_\xi | \xi < \alpha \rangle$ of terms if for every position $\overline{\eta}$ in t, there exists $\delta < \alpha$ such that for all ξ with $\delta \leq \xi < \alpha$, $\mathrm{Lab}(t_\xi, \overline{\eta}) = \mathrm{Lab}(t, \overline{\eta})$.

Rewrite rules and reductions

Let $\mathcal{F}$ be a vocabulary and X a set of variables. A *rewrite rule over $\mathcal{F}$ in X* is an ordered pair of terms s, t over $\mathcal{F}$ in variables from X such that $s \notin X$ and t contains only variables that also appear in s. We denote such a pair by $s \to t$. A pair $\langle T_{\mathcal{F}}, R \rangle$ consisting of the set of all terms over some vocabulary $\mathcal{F}$ and a set R of rewrite rules over $\mathcal{F}$ we call a *term rewriting system*, abbreviated *trs*.

A *substitution*, with respect to $\mathcal{F}$ and X, is a function $\sigma : T_{\mathcal{F},X} \to T_{\mathcal{F}}$ with the property that for every $F \in \mathcal{F}$ and every suitable sequence $\langle t_\xi | \xi < \alpha \rangle$ of arguments, $\sigma(F(t_\xi | \xi < \alpha)) = F(\sigma(t_\xi) | \xi < \alpha)$. We say $\sigma(t)$ is an *instance* of t. Now suppose we also have a set R of rewrite rules over $\mathcal{F}$. An instance of a rewrite rule $s \to t$ is a pair $\langle \sigma(s), \sigma(t) \rangle$ for some substitution σ. With reference to the trs $\langle T_{\mathcal{F}}, R \rangle$, we call such a pair a *contraction* or *head reduction*; and we write $\sigma(s) \to \sigma(t)$.

Sometimes it is useful to make the variables in a term a little more explicit. If t is a term in variables from a set $\{x_\xi | \xi < \alpha\}$, we write t as $t(x_\xi | \xi < \alpha)$. The result of substituting s_ξ for x_ξ, for all $\xi < \alpha$, is then denoted by $t(s_\xi | \xi < \alpha)$.

A *reduction* is either a contraction, or an *internal reduction*, i.e. a pair

$$\langle F(s_\xi | \xi < \alpha), \ F(t_\xi | \xi < \alpha) \rangle$$

in which for exactly one $\xi<\alpha$, $\langle s_\xi, t_\xi\rangle$ is a reduction, and for all the other ξ, $s_\xi = t_\xi$. Again, if $\langle s, t\rangle$ is a reduction, we write $s \to t$.

A reduction may be regarded as a contraction of a subterm occurrence. A pair of reductions $\{u \to v, u \to w\}$ with $v \neq w$ is a *critical pair* if u is a substitution instance $\sigma(s)$ of the lefthand side of some rewrite rule $s \to t$, $v = \sigma(t)$, and $u \to w$ contracts a subterm occurrence at a position $\overline{\eta}$ such that either Lab(s, $\overline{\eta}$) exists and is not a variable, or $\overline{\eta}$ has an initial segment $\overline{\xi}$ such that Lab(s, $\overline{\xi}$) is a variable that occurs more than once in s.

Definition 1. A *reduction sequence* is an ordinal-indexed sequence $\langle t_\xi | \xi<\alpha\rangle$ of terms such that for every $\xi<\alpha$,
if $\xi = \eta + 1$, $t_\eta \to t_\xi$;
if ξ is a limit ordinal, then t_ξ is the limit of $\langle t_\eta | \eta<\xi\rangle$.

Examples. Let F be ω-ary, S unary, and 0 and 1 nullary.
1. Suppose we have a rewrite rule $Sx \to SSx$. Then

$$(1) \qquad F(Sx, Sx,\ldots) \to F(Sx, SSx, Sx,\ldots) \to F(Sx, SSx, SSx,\ldots)$$
$$\to F(Sx, SSx, SSSx, Sx,\ldots) \to \ldots \ldots F(Sx, S^2x, S^3x, S^4x,\ldots, S^nx,\ldots)$$

is a reduction sequence. So is

$$(2) \qquad Sx \to SSx \to SSSx \to \ldots \to S^nx \to \ldots .$$

Sequence (2) does not have a limit: there is no term S^ω.
2. Suppose we have a rewrite rule $Sx \to Sx$. Then

$$Sx \to Sx \to Sx \to Sx \to \ldots \ldots$$

is a reduction sequence, with limit Sx.
3. Let $\overline{0}$ be an ω-indexed sequence of zeroes, and $\overline{1}$ a similar sequence of ones. Suppose we have rules $0 \to 1$ and for every n

$$F(x_0, \ldots, x_{n-1}, \overline{0}) \to 1.$$

Then

$$F(\overline{0}) \to F(1, \overline{0}) \to F(1, 1, \overline{0}) \to F(1, 1, 1, \overline{0}) \to \ldots\ F(\overline{1})$$

is a reduction sequence.

We write $s \twoheadrightarrow t$ if there exists a *path* from s to t, that is, a reduction sequence $\langle t_\xi | \xi<\alpha+1\rangle$ such that $t_0 = s$ and $t_\alpha = t$. We say t is a *reduct* of s, or s *reduces to* t.

We can now generalize the notion of a trs somewhat: instead of $T_{\mathcal{F}}$, for a vocabulary $\mathcal{F}$, we may take any $T \subseteq T_{\mathcal{F}}$ that is closed under subterms and reduction.

Termination and confluence

Definition 2. Let $\mathcal{R} = \langle T, R \rangle$ be a trs.
(i) A *normal form*, or *term in normal form*, is a term in T from which no reduction is possible.
(ii) A term is *weakly normalizing* if it reduces to a normal form.
(iii) A term $t \in T$ is *terminating* or *strongly normalizing* (*SN*) if
a) every reduction sequence beginning with t converges (i.e. has a limit), and
b) the reduction sequences beginning with t form a set.

Condition (b) of (iii) is equivalent to the condition that the lengths of reduction sequences from t are bounded above.

A trs $\mathcal{R}$ is called WN (SN) if every term of $\mathcal{R}$ is WN (SN). If $\mathcal{R}$ is SN, then the reduction sequences of $\mathcal{R}$ form a set; equivalently, their lengths are bounded above. Clearly, an SN term is also WN, and a fortiori a trs that is SN is also WN. Moreover, every reduct and every subterm of an SN term is SN; so if $\langle T, R \rangle$ is a trs, then so is

$$\langle \{t \in T \mid t \text{ is SN}\}, R \rangle.$$

Examples. The systems of examples 1. and 2. above are not WN, since Sx has no normal form. If we add to both a rule $Sx \to 0$, they become WN, but not SN: the first system has divergent reduction sequences, and the second has reduction sequences (repeating Sx) of any length. The system of example 3. is SN.

Definition 3. A trs $\mathcal{R} = \langle T, R \rangle$ is *confluent* (or *Church-Rosser*, abbreviated *CR*) if

(*) for all $t_1, t_2, t_3 \in T$: $t_2 \twoheadleftarrow t_1 \twoheadrightarrow t_3$ implies $\exists u \in T.\ t_2 \twoheadrightarrow u \twoheadleftarrow t_3$;

and *weakly confluent* (*weakly Church-Rosser*, *WCR*) if the condition holds that one gets from (*) by strengthening the antecedent to $t_2 \leftarrow t_1 \rightarrow t_3$.

Lemma 1. Let $\mathcal{R} = \langle T, R \rangle$ be a trs. $\mathcal{R}$ is WCR if and only if for all $t_1, t_2, t_3 \in T$:

if $t_2 \leftarrow t_1 \rightarrow t_3$ is a critical pair, then $\exists u \in T.\ t_2 \twoheadrightarrow u \twoheadleftarrow t_3$.

Proof. Straightforward; cf. lemma 3.1 in [1]. ¶

The relation $\approx$ of *convertibility* in a given trs $\mathcal{R}$ is the transitive and symmetric closure of the reduction relation $\twoheadrightarrow$. A trs $\mathcal{R}$ has the *unique normal form property* (*UN*) if for all normal forms t_1, t_2, $t_1 \approx t_2$ implies $t_1 = t_2$.

Proposition 1. For any trs,
(i) CR $\Rightarrow$ UN;
(ii) UN & WN $\Rightarrow$ CR.

Newman's Lemma for finitary reduction systems states that SN & WCR implies CR. It does not generalize to the infinitary case: the trs of example 3. above is WCR; but it is not CR, since it does not have UN. The reason why Newman's proof fails here is, of course, that our definition of SN allows infinite reduction sequences. But the present notion of strong normalization does generalize the older notion for finitary terms: by König's Lemma, an infinite sequence of terms over a finitary vocabulary that is not eventually constant must diverge.

A trs is *semi-complete* if it is WN and CR; *complete* if it is SN and CR.

Descending systems

Definition 4. Let $\mathcal{F}$ be a vocabulary, and T a set of terms over $\mathcal{F}$. A *complexity measure* for T is a well-founded partial ordering $<$ on T with the properties
(i) for any term in T of the form $F(t_\xi | \xi < \alpha)$, for every $\xi < \alpha$, if $t_\xi \in T$ then

$$t_\xi < F(t_\xi | \xi < \alpha);$$

(ii) for any α-ary $F \in \mathcal{F}$ and any α-termed sequences $\langle s_\xi | \xi < \alpha \rangle$ and $\langle t_\xi | \xi < \alpha \rangle$ of elements of T, if $s_\xi < t_\xi$ or $s_\xi = t_\xi$ for every $\xi < \alpha$, and $F(s_\xi | \xi < \alpha)$ and $F(t_\xi | \xi < \alpha)$ are distinct elements of T, then

$$F(s_\xi | \xi < \alpha) < F(t_\xi | \xi < \alpha).$$

We shall write $\leq$ for '$<$ or $=$'.

Definition 5. A trs $\mathcal{R} = \langle T, R \rangle$ over a vocabulary $\mathcal{F}$ is *weakly descending* if there exists a complexity measure for T with the property that for every contraction $t_1 \to t_2$, $t_1 > t_2$. It is *strongly descending* if $t_1 > t_2$ is ensured for every reduction $t_1 \to t_2$ that belongs to a critical pair.

It follows from definition 4 that in a weakly descending trs also $t_1 \geq t_2$ for internal reductions $t_1 \to t_2$.

Lemma 2. In a weakly descending trs, $t \twoheadrightarrow u$ implies $t \geq u$.

Proof. By induction on the complexity of t and the length of a given path from t to u. The successor step ($t \twoheadrightarrow u_0 \to u_1$, with $t \trianglerighteq u_0$) is easy. Now let u be the limit of a reduction sequence $\bar{t} = \langle t_\xi | \xi < \lambda \rangle$, with $t = t_0$. If $t_\eta \to t_{\eta+1}$ is the first head reduction in $\bar{t}$, then $t \trianglerighteq t_\eta \rhd t_{\eta+1}$, and $t_{\eta+1} \trianglerighteq u$ by induction hypothesis. So suppose all reductions in $\bar{t}$ are internal. Then for some $F \in \mathcal{F}$, every t_ξ is of the form $F(s^\gamma_\xi | \gamma < \beta)$, and $u = F(s^\gamma | \gamma < \beta)$, where $s^\gamma_0 \twoheadrightarrow s^\gamma$. By induction hypothesis, $s^\gamma_0 \trianglerighteq s^\gamma$; hence by the monotonicity condition (ii) of definition 4,

$$t = F(s^\gamma_0 | \gamma < \beta) \trianglerighteq F(s^\gamma | \gamma < \beta) = u. \qquad ¶$$

Theorem 1. Every weakly descending trs is SN.

Proof. Let $\mathcal{R} = \langle T, R \rangle$ be a weakly descending trs; suppose $\mathcal{R}$ is not SN. Then there is a term t of $\mathcal{R}$ that has minimal complexity among the terms that are not SN. Suppose $t = F(t_\xi | \xi < \alpha)$. Let β be an upper bound of the lengths of reduction sequences of $\mathcal{R}$ that begin with SN terms. Let κ be a regular initial greater than α and β. Let $\bar{t}$ be any reduction sequence beginning with t. The first head reduction in $\bar{t}$ produces a term less complex than t, so the length of the rest of $\bar{t}$ is less than κ. But the initial part of $\bar{t}$ must be composed of reductions of the t_ξ; since α is less than the cofinality of κ, the length of this part is less than κ. It follows that the total length of $\bar{t}$ is less than κ.

So the reduction sequences beginning with t form a set. Hence there must be a divergent reduction sequence $\bar{s}$ beginning with t. Since t has minimal complexity, there can be no head reductions in $\bar{s}$. So $\bar{s}$ must be of the form

$$\langle F(t^\gamma_\xi | \xi < \alpha) | \ \gamma < \delta \rangle,$$

with $t^0_\xi = t_\xi$. Now every sequence $\bar{s}_\xi = \langle t^\gamma_\xi | \ \gamma < \delta \rangle$ converges, since $t_\xi \lhd t$; but if $\bar{s}_\xi$ converges to s_ξ, for each $\xi < \alpha$, then $\bar{s}$ converges to $F(s_\xi | \xi < \alpha)$.

We conclude that there is no such term as t, which contradicts our assumption that $\mathcal{R}$ is not SN. ¶

The trs of example 3. above is weakly descending, but not strongly descending: every reduction in the example sequence belongs to a critical pair, so if these steps were decreasing we would have an infinitely decreasing sequence. We shall see that for strongly descending systems we get a variant of Newman's Lemma.

Lemma 3. Let $\mathcal{R}$ be a strongly descending trs, t a term of $\mathcal{R}$ that is a substitution instance of the righthand side of a rewrite rule $s \to s'$ and u the limit of a reduction sequence $\langle t_\xi | \xi < \alpha \rangle$ that begins with t. If $t_\xi \not< t$ for all $\xi < \alpha$, then u is a substitution instance of s.

Proof. By induction on α.

If $\alpha = \beta + 1$, then $u = t_\beta$. If $\beta = \gamma + 1$, then by induction hypothesis t_γ, the limit of $\langle t_\xi | \xi < \beta \rangle$, is of the form $\sigma(s)$. Now if t_β is not a substitution instance of s, $\sigma(s') \leftarrow t_\gamma \to t_\beta$ must be a critical pair; but then $t_\gamma > t_\beta$ (hence $t_\beta > t$) since $\mathcal{R}$ is strongly descending. If β is a limit, the conclusion is a case of the induction hypothesis.

The case remains that α is a limit. Write s as $s(x_\eta | \ \eta < \delta)$. By induction hypothesis, every t_ξ with $\xi < \alpha$ is of the form $s(w_\eta^\xi | \eta < \delta)$. Let $w_\eta = \lim_{\xi \to \alpha} w_\eta^\xi$; then $u = s(x_\eta | \ \eta < \delta)$. ¶

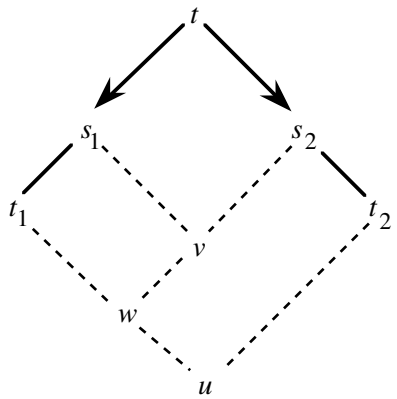
Theorem 2. If a strongly descending trs is WCR, it is confluent.

Proof. Let $\mathcal{R} = \langle T, R \rangle$ be strongly descending and WCR. Suppose $\mathcal{R}$ is not confluent. Then there exist terms that reduce to terms without common reduct; let t be minimal with this property. We have

$$t_1 \twoheadleftarrow t \twoheadrightarrow t_2, \text{ and for no } u \in T,\ t_1 \twoheadrightarrow u \twoheadleftarrow t_2.$$

Fix paths from t to t_1 and from t to t_2. We consider three cases (leaving some degenerate cases to the reader).

(1) The first steps of $t \twoheadrightarrow t_1$ and $t \twoheadrightarrow t_2$ (say $s_1 \leftarrow t \to s_2$) form a critical pair. By WCR, s_1 and s_2 have a common reduct v. Now since $s_1 < t$, and $t_1 \twoheadleftarrow s_1 \twoheadrightarrow v$, t_1 and v have a common reduct w. But similarly t_2 and w have a common reduct u. So this case cannot apply.

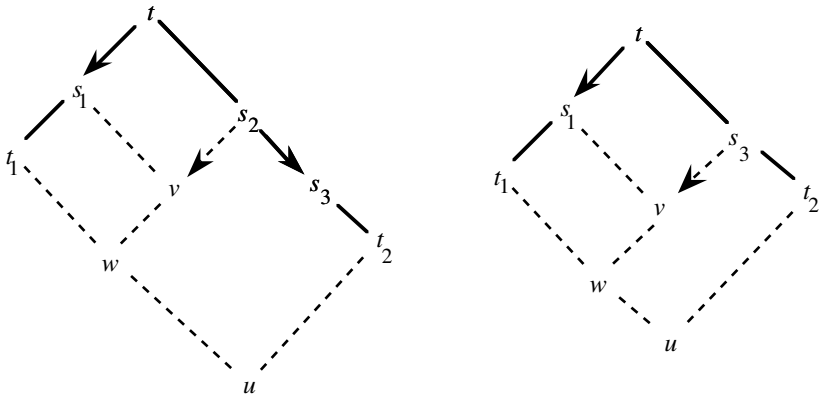


(2) One path begins with a contraction, and the other with a sequence of internal reductions that do not reduce complexity. Say we have

$$t_1 \twoheadleftarrow s_1 \leftarrow t \twoheadrightarrow s_2 \rightarrow s_3 \twoheadrightarrow t_2,$$

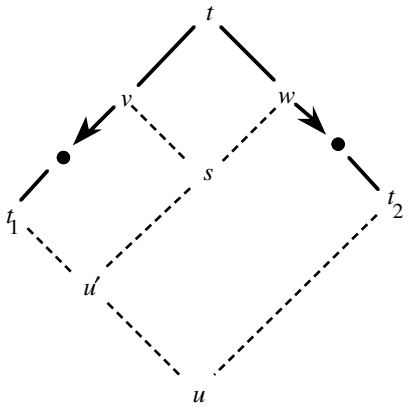
where $t \rightarrow s_1$ is a contraction, and s_3 is the first term s on the path from t to t_2 with $s \lhd t$. There are two subcases: either (a) s_3 has an immediate predecessor s_2 on the path $t \twoheadrightarrow t_2$, or (b) it has not. Let $r_1 \rightarrow r_2$ be the rule applied in $t \rightarrow s_1$. Suppose $r_1 = r_1(x_\xi|\xi<\alpha)$, $r_2 = r_2(x_\xi|\xi<\alpha)$. Then we may write t as $r_1(t_\xi|\xi<\alpha)$, and $s_1 = r_2(t_\xi|\xi<\alpha)$.

(a) By lemma 3, s_2 is of the form $r_1(s_\xi|\xi<\alpha)$, with $t_\xi \twoheadrightarrow s_\xi$. Let $v := r_2(s_\xi|\xi<\alpha)$; then v is a common reduct of s_1 and s_2. Since $s_1 \lhd t$, and $t_1 \twoheadleftarrow s_1 \twoheadrightarrow v$, t_1 and v have a common reduct w. Finally, by case (1) t_2 and w have a common reduct u. So this subcase cannot apply.



(b) Since s_3 is a limit, by lemma 3 it is of the form $r_1(s_\xi|\xi<\alpha)$, with $t_\xi \twoheadrightarrow s_\xi$. Let $v := r_1(s_\xi|\xi<\alpha)$. Then v is a common reduct of s_1 and s_3. As before, t_1 and v have a common reduct w. And since $s_3 \lhd t$, t_2 and w have a common reduct u. This eliminates case (2).

(3) Both paths begin with internal reductions. Say we have $v \twoheadleftarrow t \twoheadrightarrow w$, $t = F(t_\xi|\xi<\alpha)$, $v = F(v_\xi|\xi<\alpha)$, and $w = F(w_\xi|\xi<\alpha)$. Since $t_\xi \lhd t$, each v_ξ and w_ξ have a common reduct s_ξ; so v and w have a common reduct $s = F(s_\xi|\xi<\alpha)$. This suffices: either we get a common reduct of t_1 and t_2, or we are back in an earlier case. Thus in the figure below, $t_1 \twoheadrightarrow u' \twoheadleftarrow s$, and hence $u_1 \twoheadrightarrow u \twoheadleftarrow t_2$, by (1) or (2).



We have shown that t cannot exist; hence $\mathcal{R}$ must be confluent. ¶

Corollary. A strongly descending trs that is WCR is complete.

Example: abstract clones over a many-sorted signature

Many-sorted signatures. Let S be a set. A *function declaration over* S is a triple $\mathbf{f} = \langle \mathrm{f}, \mathrm{s}, \langle \mathrm{s}_i | \ i \in I \rangle \rangle$ consisting, in order, of some thing f (a function symbol), an element of S, and a family of elements of S.[1] A (many-sorted) *signature* is a set $\Sigma = \mathrm{S} \cup \mathrm{F}$ where F is a set of function declarations over S. We call S the set of *sort symbols* (or *sorts*) of Σ; we write $\mathrm{S}\Sigma$ for S and $\mathrm{F}\Sigma$ for F. The second and third elements of a function declaration $\mathbf{f}$ are called its *sort* and its *argument type*; the ordered pair of these elements is the *type of* $\mathbf{f}$.

Objects. Let Σ be a signature, and U a set that *admits* Σ, that is, contains the index sets of the argument types of Σ. Then the set $\mathrm{Ob} = \mathrm{Ob}(\Sigma, U)$ of *objects* over Σ and U is inductively defined as follows:
(i) $\mathrm{S}\Sigma \subseteq \mathrm{Ob}(\Sigma, U)$;
(ii) if $I \in U$ and $\boldsymbol{k}$ is an I-indexed family of elements of $\mathrm{Ob}(\Sigma, U)$, then $\boldsymbol{k} \in \mathrm{Ob}(\Sigma, U)$.

The intuition behind (ii) is, that certain families of objects have products. In the formal setting, there is no need to distinguish such families from their products.

Expressions. The set $\mathrm{Exp} = \mathrm{Exp}(\Sigma, U)$ of *expressions* over Σ and U is defined inductively together with operations dom (domain) and cod (codomain) from Exp to Ob, as follows:
(i) If $\mathbf{f} = \langle \mathrm{f}, \mathrm{s}, \boldsymbol{s} \rangle \in \mathrm{F}\Sigma$, then $\mathbf{f} \in \mathrm{Exp}$; $\mathrm{dom}\mathbf{f} = \boldsymbol{s}$ and $\mathrm{cod}\mathbf{f} = \mathrm{s}$.
(ii) If $\boldsymbol{k} \in \mathrm{Ob}$, then the identity $1_{\boldsymbol{k}}$ belongs to Exp.
(iii) If $\boldsymbol{k} = \langle k_i | \ i \in I \rangle \in \mathrm{Ob}$, then the *formal variable* $\mathbf{x}_i^{\boldsymbol{k}}$ is in Exp; $\mathrm{dom}\mathbf{x}_i^{\boldsymbol{k}} = \boldsymbol{k}$ and $\mathrm{cod}\mathbf{x}_i^{\boldsymbol{k}} = k_i$.
(iv) If $\mathbf{e}_1, \mathbf{e}_2 \in \mathrm{Exp}$ and $\mathrm{dom}(\mathbf{e}_1) = \mathrm{cod}(\mathbf{e}_2)$, then $\mathbf{e}_1 \circ \mathbf{e}_2 \in \mathrm{Exp}$;
$\mathrm{dom}(\mathbf{e}_1 \circ \mathbf{e}_2) = \mathrm{dom}(\mathbf{e}_2)$ and $\mathrm{cod}(\mathbf{e}_1 \circ \mathbf{e}_2) = \mathrm{cod}(\mathbf{e}_1)$.
(v) If $I \in U$ and $\langle \mathbf{e}_i | \ i \in I \rangle$ is an I-indexed family of elements of Exp with common domain k, then $\boldsymbol{e} = (\mathbf{e}_i | \ i \in I) \in \mathrm{Exp}$; $\mathrm{dom}(\boldsymbol{e}) = k$ and

$$\mathrm{cod}(\boldsymbol{e}) = \langle \mathrm{cod}(\mathbf{e}_i) | \ i \in I \rangle.$$

[1] We assume that none of the elements of S are triples, in order to avoid confusion with function declarations.

Exp may be considered a set of terms in the sense used above: the function declarations, identities, and formal variables are constants, $\circ$ is a binary function symbol, and if we fix a well-ordering for every $I \in U$, we may take (—), in association with I, to be an α-ary function symbol, where α is the type of the well-ordering of I.

Consider the following system $R = R_{\Sigma,U}$ of rule schemes for rewriting expressions, in which **c**, **d**, and **e**, and their indexed variants, are to be regarded as variables:

$$\mathbf{x}_j^{\boldsymbol{k}} \circ (\mathbf{e}_i |\ i \in I) \to \mathbf{e}_j$$

$$(\mathbf{e}_i |\ i \in I) \circ \mathbf{d} \to (\mathbf{e}_i \circ \mathbf{d} |\ i \in I)$$

$$(\mathbf{c} \circ \mathbf{d}) \circ \mathbf{e} \to \mathbf{c} \circ (\mathbf{d} \circ \mathbf{e})$$

$(\mathbf{x}_i^{\boldsymbol{k}} \circ \mathbf{e} |\ i \in I) \to \mathbf{e}$, if $\boldsymbol{k}$ is of the form $\langle k_i |\ i \in I \rangle$

$(\mathbf{x}_i^{\boldsymbol{k}} |\ i \in I) \to 1_{\boldsymbol{k}}$, if $\boldsymbol{k}$ is of the form $\langle k_i |\ i \in I \rangle$

$$1_{\boldsymbol{k}} \circ \mathbf{e} \to \mathbf{e}$$

$$\mathbf{e} \circ 1_{\boldsymbol{k}} \to \mathbf{e}$$

It is easily checked that Exp is closed under these rules; so $\langle \text{Exp}, R \rangle$ is a trs.

The *complexity* of an expression **e** is the ordinal $|\mathbf{e}|$ defined as follows:
$|\mathbf{x}_i^{\boldsymbol{k}}| = |1_{\boldsymbol{k}}| = |\mathbf{f}| = 3$, for all $\boldsymbol{k}$ and i, and $\mathbf{f} \in \text{F}\Sigma$;
$|\mathbf{d} \circ \mathbf{e}| = (|\mathbf{d}| \cup |\mathbf{e}|)^{|\mathbf{d}|}$;
$|(\mathbf{e}_i |\ i \in I)| = \bigcup_{i \in I} |\mathbf{e}_i| + 4$.

Lemma 4. The relation $|\mathbf{d}| < |\mathbf{e}|$ is a complexity measure on Exp.

Proof. An exercise in ordinal arithmetic. Use the fact that for any ordinal $\xi > 1$, $\xi^1 = \xi$, and $\alpha < \beta$ implies $\alpha^\xi \le \beta^\xi$ and $\xi^\alpha < \xi^\beta$. ¶

Lemma 5. The system $\langle \text{Exp}, R \rangle$ is strongly descending.

Proof. A longer exercise in ordinal arithmetic. We check the least trivial cases, contractions first.
(i) $|(\mathbf{e}_i \circ \mathbf{d} |\ i \in I)| > |(\mathbf{e}_i |\ i \in I) \circ \mathbf{d}|$: let $\varepsilon = \bigcup_{i \in I} |\mathbf{e}_i|$. Then since $|\mathbf{d}| \cup (\varepsilon + 4) \ge 4$,

$$|(\mathbf{e}_i \circ \mathbf{d} |\ i \in I)| = \bigcup_{i \in I} (|\mathbf{e}_i| \cup |\mathbf{d}|)^{|\mathbf{e}_i|} + 4 \le ((\varepsilon + 4) \cup |\mathbf{d}|)^\varepsilon + 4$$

$$< ((\varepsilon + 4) \cup |\mathbf{d}|)^{\varepsilon + 4} = |(\mathbf{e}_i |\ i \in I) \circ \mathbf{d}|$$

(ii) $|(\mathbf{c} \circ \mathbf{d}) \circ \mathbf{e}| > |\mathbf{c} \circ (\mathbf{d} \circ \mathbf{e})|$: put $\alpha = |\mathbf{c} \circ (\mathbf{d} \circ \mathbf{e})|$ and $\beta = |(\mathbf{c} \circ \mathbf{d}) \circ \mathbf{e}|$. Then

$$\alpha = (|\mathbf{c}| \cup |\mathbf{d} \circ \mathbf{e}|)^{|\mathbf{c}|} = ((|\mathbf{d}| \cup |\mathbf{e}|)^{|\mathbf{d}|} \cup |\mathbf{c}|)^{|\mathbf{c}|}$$

and

$$\beta = (|\mathbf{e}| \cup |\mathbf{c} \circ \mathbf{d}|)^{|\mathbf{c} \circ \mathbf{d}|} = (|\mathbf{e}| \cup (|\mathbf{c}| \cup |\mathbf{d}|)^{|\mathbf{c}|})^{(|\mathbf{c}| \cup |\mathbf{d}|)^{|\mathbf{c}|}}.$$

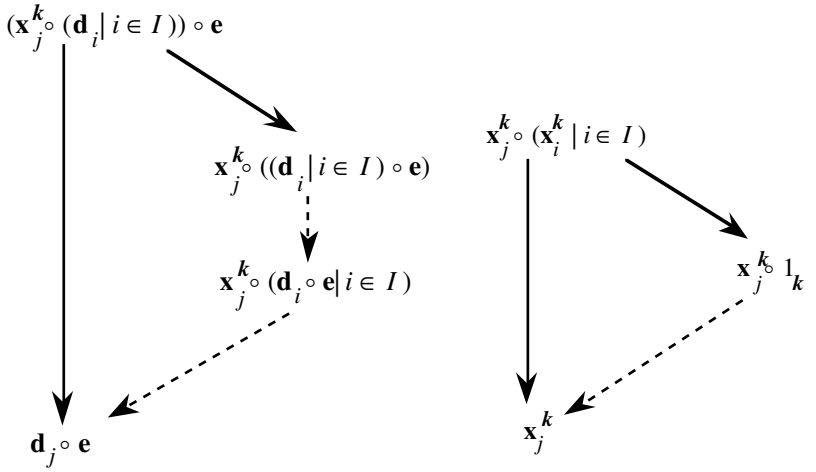
Now we distinguish cases according to the relative sizes of $|\mathbf{c}|$, $|\mathbf{d}|$, and $|\mathbf{e}|$.
$|\mathbf{c}| \le |\mathbf{d}| \le |\mathbf{e}|$: $\alpha = (|\mathbf{e}|^{|\mathbf{d}|} \cup |\mathbf{c}|)^{|\mathbf{c}|} = |\mathbf{e}|^{|\mathbf{d}| \cdot |\mathbf{c}|}$, and $\beta = (|\mathbf{e}| \cup |\mathbf{d}|^{|\mathbf{c}|})^{|\mathbf{d}|^{|\mathbf{c}|}}$. Now $|\mathbf{d}| \cdot |\mathbf{c}| \le |\mathbf{d}|^2 < |\mathbf{d}|^{|\mathbf{c}|}$ since $|\mathbf{c}| > 2$, hence $\alpha < \beta$.
$|\mathbf{c}| \le |\mathbf{e}| \le |\mathbf{d}|$: $\alpha = (|\mathbf{d}|^{|\mathbf{d}|} \cup |\mathbf{c}|)^{|\mathbf{c}|} = |\mathbf{d}|^{|\mathbf{d}| \cdot |\mathbf{c}|}$, and $\beta = (|\mathbf{e}| \cup |\mathbf{d}|^{|\mathbf{c}|})^{|\mathbf{d}|^{|\mathbf{c}|}} = |\mathbf{d}|^{|\mathbf{c}| \cdot |\mathbf{d}|^{|\mathbf{c}|}}$. Now since $|\mathbf{d}| < |\mathbf{d}|^{|\mathbf{c}|}$ and $|\mathbf{d}| \cdot |\mathbf{c}| < |\mathbf{d}|^{|\mathbf{c}|}$, $\alpha < \beta$.
$|\mathbf{d}| \le |\mathbf{c}| \le |\mathbf{e}|$: $\alpha = (|\mathbf{e}|^{|\mathbf{d}|} \cup |\mathbf{c}|)^{|\mathbf{c}|} = |\mathbf{e}|^{|\mathbf{d}| \cdot |\mathbf{c}|}$, and $\beta = (|\mathbf{e}| \cup |\mathbf{c}|^{|\mathbf{c}|})^{|\mathbf{c}|^{|\mathbf{c}|}}$. Use: $|\mathbf{d}| \cdot |\mathbf{c}| \le |\mathbf{c}|^2 < |\mathbf{c}|^{|\mathbf{c}|}$.
$|\mathbf{d}| \le |\mathbf{e}| \le |\mathbf{c}|$: $\alpha = (|\mathbf{e}|^{|\mathbf{d}|} \cup |\mathbf{c}|)^{|\mathbf{c}|}$, and $\beta = |\mathbf{c}|^{|\mathbf{c}| \cdot |\mathbf{c}|^{|\mathbf{c}|}}$. Use that $|\mathbf{e}|^{|\mathbf{d}|} \cup |\mathbf{c}| < |\mathbf{c}|^{|\mathbf{c}|}$.
$|\mathbf{e}| \le |\mathbf{c}| \le |\mathbf{d}|$: $\alpha = |\mathbf{d}|^{|\mathbf{d}| \cdot |\mathbf{c}|}$, and $\beta = |\mathbf{d}|^{|\mathbf{c}| \cdot |\mathbf{d}|^{|\mathbf{c}|}}$. Then $\alpha < \beta$ since $|\mathbf{d}|, |\mathbf{d}| \cdot |\mathbf{c}| < |\mathbf{d}|^{|\mathbf{c}|}$.
$|\mathbf{e}| \le |\mathbf{d}| \le |\mathbf{c}|$: $\alpha = (|\mathbf{d}|^{|\mathbf{d}|} \cup |\mathbf{c}|)^{|\mathbf{c}|}$, and $\beta = |\mathbf{c}|^{|\mathbf{c}| \cdot |\mathbf{c}|^{|\mathbf{c}|}}$. Then $\alpha < \beta$ since both $|\mathbf{d}|^{|\mathbf{d}|}$ and $|\mathbf{c}|$ are less than $|\mathbf{c}|^{|\mathbf{c}|}$.
(iii) $|(\mathbf{x}_i^k \mid i \in I)| > |1_k|$, even if $I = \varnothing$.
(iv) If $|\mathbf{d}'| < |\mathbf{d}|$, then $|\mathbf{d}' \circ \mathbf{e}| < |\mathbf{d} \circ \mathbf{e}|$.
(v) $|\mathbf{x}_j^k \circ (\mathbf{x}_i^k \circ \mathbf{e} \mid i \in I)| = (3 \cup ((3 \cup |\mathbf{e}|)^3 + 4)) > (3 \cup |\mathbf{e}|)^3 = |\mathbf{x}_j^k \circ \mathbf{e}|$. ¶

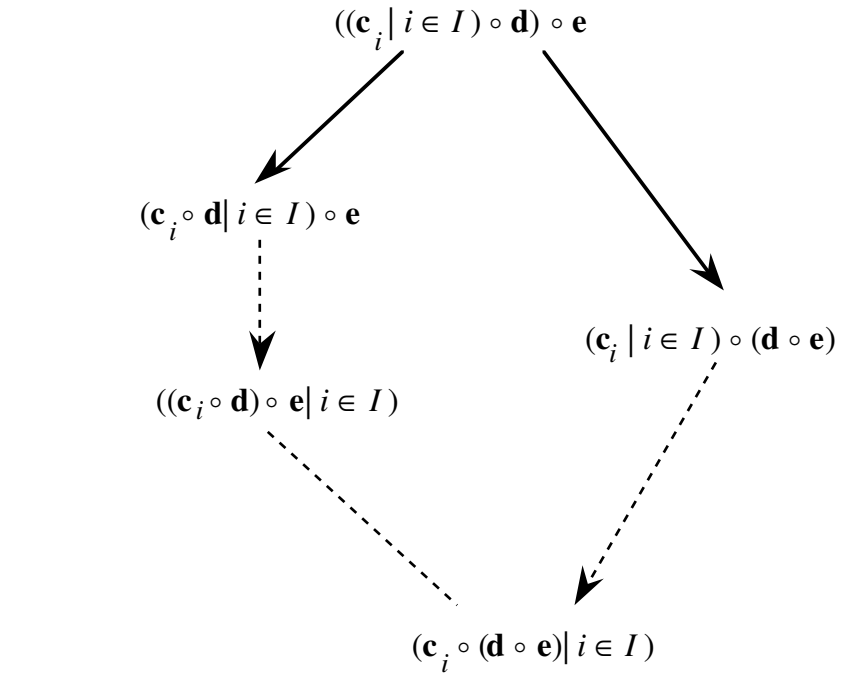
Proposition 2. $\langle \mathrm{Exp}, R \rangle$ is SN.

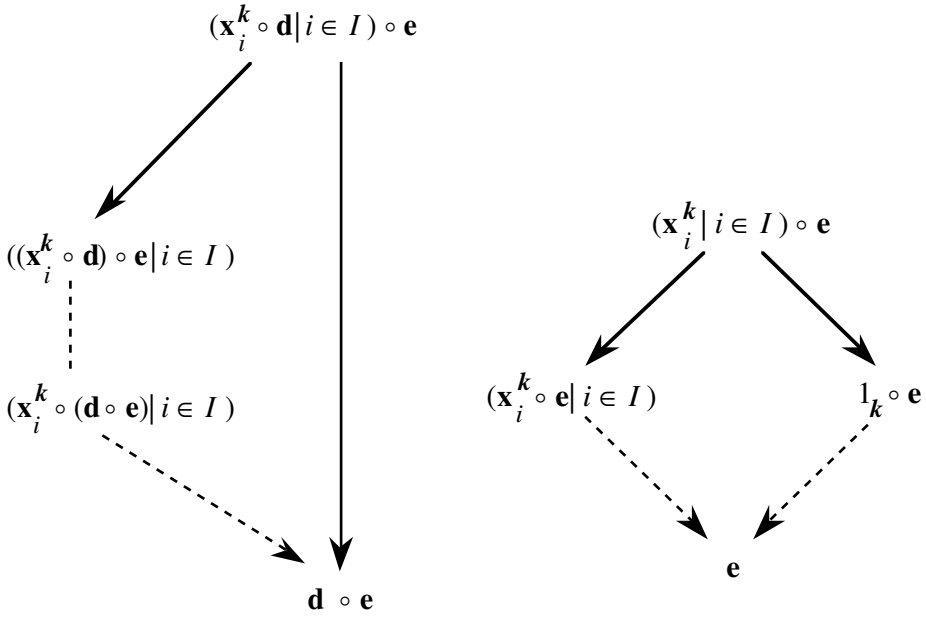
Proof. Immediate by theorem 1 and lemma 5 above. ¶

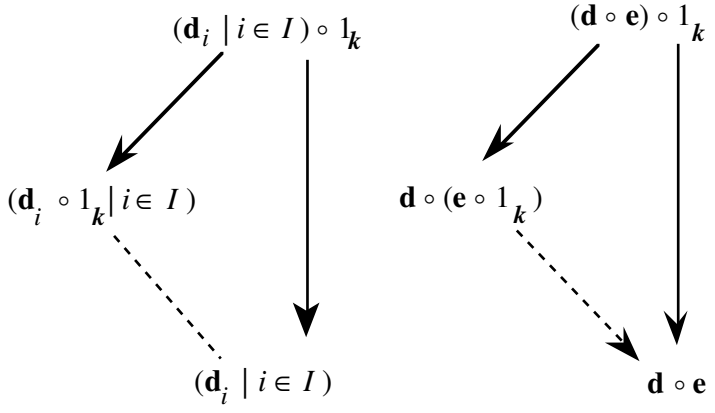
Lemma 6. $\langle \mathrm{Exp}, R \rangle$ is WCR.

Proof. Given a critical pair $t_2 \leftarrow t_1 \rightarrow t_3$, we are to find a common reduct u of t_2 and t_3 (lemma 1). The manner is shown in the diagrams below.




((c_i | i ∈ I) ∘ d) ∘ e
(c_i ∘ d | i ∈ I) ∘ e
(c_i | i ∈ I) ∘ (d ∘ e)
((c_i ∘ d) ∘ e | i ∈ I)
(c_i ∘ (d ∘ e) | i ∈ I)


(x_i^k ∘ d | i ∈ I) ∘ e
((x_i^k ∘ d) ∘ e | i ∈ I)
(x_i^k ∘ (d ∘ e) | i ∈ I)
d ∘ e
(x_i^k | i ∈ I) ∘ e
(x_i^k ∘ e | i ∈ I)
1_k ∘ e
e


(d_i | i ∈ I) ∘ 1_k
(d_i ∘ 1_k | i ∈ I)
(d_i | i ∈ I)
(d ∘ e) ∘ 1_k
d ∘ (e ∘ 1_k)
d ∘ e

¶

Proposition 3. $\langle \mathrm{Exp}, R \rangle$ is confluent.

Proof. Immediate by Theorem 2 and Lemma 4 above. ¶

Corollary. $\langle \mathrm{Exp}, R \rangle$ is complete.

It follows that the expressions in normal form constitute the arrow set of a Cartesian category where e.g. the composite of arrows **d** and **e** is the normal form of **d**∘**e**. In particular, it is immediately clear that distinct terms over Σ, in the usual sense of many-sorted algebra, correspond with distinct elements of the abstract clone.

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