Network Algebra with Demonic Relation Operators

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Abstract

Relations with demonic operators are used in studies related to predicate transformer semantics of nondeterministic programs, to model the connection wires in synchronous dataflow networks, or in process algebra setting to model the communication between agents and their envirnoments.

The aim of this paper is to provide equational axiomatisations for various classes of finite relations with demonic (including looping) operators. We present axiomatisations for three types of demonic calculi: i.e. calculi for relations with forward demonic –, backward demonic – or two-way demonic operators. The algebraic structures involved are basic network algebras (a certain class of symmetric strict monoidal categories enriched with a looping operation – feedback) with appropriate ramification and identification constants.

Keywords & Phrases: network algebra, relations, demonic calculus, synchronous dataflow networks, feedback.

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1 Introduction

The algebraic theory of relations has a long tradition, see e.g. [ScS93, BrS95]. Most of the work has been done in an homogeneous setting to study relations over a set. When heterogeneous relations have been considered they where usually modelled as matrices over an appropriate semiring. However, the semiring structure turned out to be quite restrictive with its distributivity and cancelation laws and to narrow the area of applications of such algebraic studies. The representation of relations as enriched symmetric strict monoidal categories (ssmc's, for short) emerged as a possible solution in the settings where the duplication or cancelation of processes are problematic.

One point of debate in such algebraic calculi for relations is the definition of (sequential) composition. Most of the work has been done using the "angelic" view, i.e. the situation when a pair (x, z) is in the composite relation $R \circ S$ if *there exists* an intermediary element y such that $(x, y) \in R$ and $(y, z) \in S$.

Axiomatisations as enriched ssmc's for various classes of finite relations under angelic composition are presented in [CaS91]. The operations and the constants used in [CaS91] are: parallel composition, (angelic) sequential composition and six types of constants, i.e. identity (I), transposition (X), identification ($\top, \vee$) and ramification ($\perp, \wedge$) constants. These axiomatisations were further extended to a setting including a looping operator (feedback $\uparrow$) in order to be used in the calculus of flownomials, cf. [CaS90, Ste94].

The aim of the present paper is to provide equational axiomatisations for classes of finite relations using demonic operators. Roughly speaking, the demonic view consist in the slogan:

> If in a branching point an input (resp. output) disconnected path exists, then it destroyes the other input (resp. output) connected paths of that point.

By making distinction between incomming paths, outgoing paths, or both we get three types of demonic calculi, i.e. forward demonic -, backward demonic -, and two-way demonic calculus, respectively.

Relations with backward demonic operators have been used to study predicate transformer semantics for nondeterministic programs, see e.g. [DMN95]. In this theory one is interested to get the weakest liberal precondition $\mathtt{wlp}(P, \phi)$ which guarantees the termination of the program P in a state fulfilling property ϕ. Now, if one looks, e.g. at the sequential composite of two programs $P \cdot Q$, then a state s guarantees the termination of $P \cdot Q$ iff s guarantees the termination of P *and* in all states which are reached by P starting from s the termination of Q is guaranteed. Using the terms of the above slogan one may say that a path in P starting from a state s which fails to be forwarded by a terminating path in Q destroyes the other paths of P from s.

Our interest for the axiomatisation of relations with demonic operators has started with the forward demonic case which may be used to model the connection wires in synchronous dataflow networks [BeMS94b] (SDFNs, for short). In the axiomatic point of view of the present paper this means the operations we are using are parallel and forward demonic sequential composition and forward demonic feedback.

It turned out that a slight change of the axioms in [CaS91, Ste94] was needed to provide a correct and complete axiomatisation for relations with forward demonic operators, i.e. the originary axioms A3 and F5 have been replaced by A3^o and F5^o, respectively.

We use this axiomatisation of relations with forward demonic operators to study the algebraic theory of synchronous dataflows [BeMS94b]. This model consists of networks of cells that compute and communicate data in an synchronous way. Such a network has a (local) crash property. That is, if one component fails to deliver any of its output data, an undefined datum results and it destroyes the other data it will be engaged in computation with. So using forward demonic relation operators one can determine whether or not a SDFN is correct in the sense that it transforms a vector of input streams into a vector of output streams. In this way forward demonic operators occur in modeling synchronous dataflow networks.

Other work using demonic models exists. We mention [Jos92, dNS95]. In [Jos92] the author reworks Hoare's CSP with a particular demonic inetrpretation of the communication. In [dNS95] a (CCS-like) process algebra model DIOA for demonic input/output automata is presented and axiomatisations for recursion-free processes are given.

2 Definitions, examples

Some graphical algorithms may ease the computations in the algebraic calculi we shall define for relations with angelic, forward demonic, backward demonic, or two-way demonic operators. In general, a relational term $f : m \to n$ is represented as a graph

$$G(f) = (I, O, Int; E)$$

where $I = \{in_1, \ldots, in_m\}$ (resp. $O = \{out_1, \ldots, out_n\}$; resp. Int) is the set of input (resp. output; resp. internal) vertices of $G(f)$, and E is the set of the edges of $G(f)$. We suppose I, O and Int are disjoint sets.

Notice that in the angelic case an input and an output of a relational term f are connected in the resulting relation iff there exists a path in $G(f)$ connecting them. On the oposite extreme, in the two-way demonic case a whole connected component containing an input or output vertex in $G(f)$ is destroyed if that vertex belong to a maximal path which do not connect an input with an output. The forward demonic case is in between and in this case all the paths containing an output are destroyed if that vertex belong to a maximal path which fails to start from an input. The situations with backward demonic operators is similar.

Graphical representation of relational terms:

A relation $f \subseteq [m]\times[n]$ is graphically represented using m input vertices $in_1, \ldots, in_m$ and n output vertices $out_1, \ldots, out_n$ (and no internal vertices) by

$$G(f) = (\{in_1, \ldots, in_m\}, \{out_1, \ldots, out_n\}, \emptyset : \quad \{(in_i, out_j) : (i,j) \in f\})$$

For instance, the constants may be represented as

$$G(\mathsf{I}_1) = (\{in_1\}, \{out_1\}, \emptyset; \quad \{(in_1, out_1)\})$$

$$G({}^1\mathsf{X}^1) = (\{in_1, in_2\}, \{out_1, out_2\}, \emptyset; \quad \{(in_1, out_2), (in_2, out_1)\})$$

$$G(\wedge_k^1) = (\{in_1\}, \{out_1, \ldots, out_k\}, \emptyset; \quad \{(in_1, out_1), \ldots, (in_1, out_k)\})$$

$$G(\vee_1^k) = (\{in_1, \ldots, in_k\}, \{out_1\}, \emptyset; \quad \{(in_1, out_1), \ldots, (in_k, out_1)\})$$

In praticular, this way we get graphical representations for the constants $\perp^1 \ (= \wedge_0^1), \quad \wedge^1 \ (= \wedge_2^1), \quad \top_1 \ (= \vee_1^0), \quad \vee_1 \ (= \vee_1^2)$.

Let $t : m \to n$ be a relational term over $\mathsf{I}, \mathsf{X}, \perp, \wedge, \top, \vee, +\!\!+, \circ, \uparrow$, where $\circ/\uparrow$ generically denotes the pair of sequential composition/feedback operators used in one of the angelic, forward demonic, backward demonic, or two-way demonic calculus. Then t is represented as a graph $G(t)$ starting from the graphical representation of the constants and using the following rules for the operators:

- $G(f +\!\!+ g)$, for $f : m \to n$, $g : p \to q$, is obtained renaming the labels of the input (resp. output) vertices in $G(g)$ into $in_{m+1}, \ldots, in_{m+p}$ (resp. into $out_{n+1}, \ldots, out_{n+q}$) and making the disjoint union with $G(f)$.

- $G(f \circ g)$ is the graph obtained identifying the outputs of $G(f)$ with the inputs of $G(g)$; these vertices become internal vertices in $G(f \circ g)$.

- $G(f \uparrow^1)$ for $f : m+1 \to n+1$ is obtained from $G(f)$ by adding an edge from out_{n+1} to in_{m+1} and transforming in_{m+1} and out_{n+1} into internal vertices. □

A maximal path in $G(f)$ is call *i-disconnected* if it fails to start with an input. That is, an i-disconnected path is either infinite (say, it is *looping*) or starts from an internal vertex with no incoming edges (say, it is *not looping*). Similarly, an *o-disconnected* path is a maximal path in $G(f)$ not ending in an output vertex. Finally, a *connected component* in $G(f)$ is just a connected component in the graph obtained making undirected all the edges of $G(f)$. [1]

Algorithm: Suppose $t : m \to n$ is a relational term and $G(t)$ is the corresponding graph obtained using the above rules. Then:

1. the resulting relation $\mathbf{a}(G(f))$ computed in the *angelic* case is

$$\mathbf{a}(G(f)) = \{(in_i, out_j) : \text{there is a path } in_i \to \ldots \to out_j \text{ in } G(f)\}$$

2. the resulting relation $\mathbf{fd}(G(f))$ computed in the *forward demonic* case is $\mathbf{a}(G_o(f))$, where $G_o(f)$ is obtainig from $G(f)$ by deleting all i-disconnected paths.

3. the resulting relation $\mathbf{bd}(G(f))$ computed in the *backward demonic* case is $\mathbf{a}(G_i(f))$, where $G_i(f)$ is obtainig from $G(f)$ by deleting all o-disconnected paths.

(iv) finally, the resulting relation $\mathbf{fbd}(G(f))$ computed in the *two-way demonic* case is $\mathbf{a}(G_{io}(f))$, where $G_{io}(f)$ is the graph obtaining from $G(f)$ by iteratively deleting all i-disconnected or o-disconnected paths as long as this is possible.[2] □

[1] Hence, the connected component which contains a vertex v consists of all the vertices which may be reached by paths starting from v and using the arrows in both directions.

[2] It may be seen that $G_{io}(f)$ is obtained by removing a connected component in $G(f)$ if it contains an i- or o-disconnected path.

Example 2.1 For an illustration of the algorithm we consider the relational term

$$f = (\wedge^1 +\!\!+ \wedge^1 +\!\!+ \mathsf{I}_1 +\!\!+ \perp^1) \circ (\mathsf{I}_1 +\!\!+ \vee_1 \circ (\top_1 +\!\!+ \mathsf{I}_1) \circ \vee_1 +\!\!+ \vee_1 \circ [(\wedge^2 \circ (\mathsf{I}_2 +\!\!+ \vee_1)) \uparrow^1])$$

Its graphical representation $G(f)$ is illustrated in Figure 1. Using the algorithm one can easily compute the resulting relation obtained in various calculi, namely:

angelic calculus	$\{(in_1, out_1), (in_1, out_2), (in_2, out_2), (in_2, out_3),$ $(in_2, out_4), (in_3, out_3), (in_3, out_4)\}$
forward demonic calculus	$\{(in_1, out_1), (in_2, out_3), (in_3, out_3)\}$
backward demonic calculus	$\{(in_1, out_1), (in_1, out_2)\}$
two-way demonic calculus	$\emptyset$

A few explanations may be useful. To compute the result obtained in the forward demonic case first we detect the maximal i-disconnected paths

$$\begin{array}{rl} 1 \to out_2 & \text{(not looping, i-disconnected path)} \\ \ldots \to \underline{6 \to 0 \to 9} \to \underline{6 \to 0 \to 9} \to 4 \to out_4 & \text{(looping, i-disconnected path)} \end{array}$$

Hence all the tuples obtained in the angelic case which contain out_2 or out_4 are deleted and the result follows.

Similarly for the backward demonic case.

In the two-way demonic case, in a first step we delete the maximal i- or o-disconnected paths

$$\begin{array}{rl} 1 \to out_2 & \text{(not looping, i-disconnected path)} \\ \ldots \to \underline{6 \to 0 \to 9} \to \underline{6 \to 0 \to 9} \to 4 \to out_4 & \text{(looping, i-disconnected path)} \\ in_2 \to 13 \to 8 \to 5 \to \underline{0 \to 9 \to 6} \to \underline{0 \to 9 \to 6} \to \ldots & \text{(looping, o-disconnected path)} \\ in_3 \to 14 \to 8 \to 5 \to \underline{0 \to 9 \to 6} \to \underline{0 \to 9 \to 6} \to \ldots & \text{(looping o-disconnected path)} \end{array}$$

At the second step new maximal disconnected paths occur

$$\begin{array}{rl} 3 \to out_3 & \text{(not looping, i-disconnected path)} \\ in_1 \to 11 \to 7 \to 2 & \text{(not looping, o-disconnected path)} \end{array}$$

which are deleted. Finally, at the third step the path

$$10 \to out_1 \quad \text{(not looping, i-disconnected path)}$$

is deleted and the empty relation result is obtained. $\Box$

3 Acyclic forward demonic case

We denote by REL^{fd} the algebraic structure of finite relations

$$\mathsf{REL}^{fd}(m, n) = \{r : \ r \subseteq [m] \times [n]\}$$

The basic operations are:

- **parallel composition** $+\!\!+$:
  if $f : m \to n$ and $g : p \to q$, then $f +\!\!+ g : m + p \to n + q$ is the relation

$$f +\!\!+ g = f \ \cup \ \{(m + i, n + j) : \ (i, j) \in g\}$$

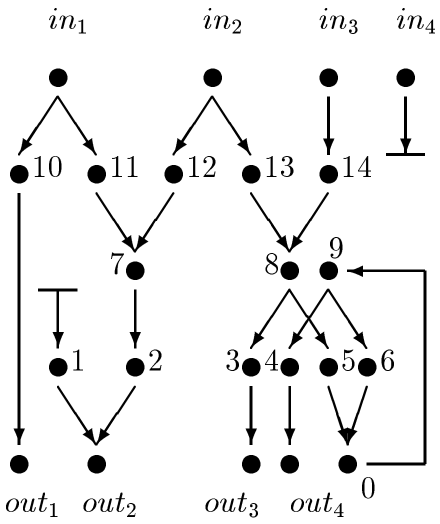

Figure 1: The graphical representation of a relational term

- **forward demonic sequential composition** $\circ_{\mathrm{fd}}$:
  if $f : m \to n$ and $g : n \to p$, then $f \circ_{\mathrm{fd}} g : m \to p$ is the relation defined by

$$f \circ_{\mathrm{fd}} g \quad = \quad \{\ (i,j) : \quad \begin{array}{l} \exists k \in [n].\ \ [(i,k) \in f\ \&\ (k,j) \in g] \quad \text{and} \\ \forall k' \in [n].\ \ [(k',j) \in g\ \Rightarrow\ \exists i' \in [m].(i',k') \in f]\ \} \end{array}$$

We recall that the usual (angelic) sequential composition is defined by

$$f \cdot g \quad = \{\ (i,j) : \ \exists k \in [n].\ \ [(i,k) \in f\ \&\ (k,j) \in g]\ \}$$

for f, g as above, hence the forward demonic sequential composition is included in the angelic sequential composition. In the forward demonic case, a connection from in_i to out_j in $f \cdot g$ is a connection in $f \circ_{\mathrm{fd}} g$, as well, precisely in the case all the maximal paths in $G(f \cdot g)$ ending in out_j starts from an input in_k, for a $k \in [m]$.

Besides these operations we are using the following constants

identity $\mathsf{I}_1 : 1 \to 1$ defined by: $\mathsf{I}_1 = \{(1,1)\}$

transposition ${}^1\mathsf{X}^1 : 2 \to 2$ defined by: ${}^1\mathsf{X}^1 = \{(1,2),(2,1)\}$

ramification $\wedge^1_k : 1 \to k$ for $k \geq 0$ defined by: $\wedge^1_k = \{(1,1),\ldots,(1,k)\}$

identification $\vee^k_1 : k \to 1$ for $k \geq 0$ defined by: $\vee^k_1 = \{(1,1),\ldots,(k,1)\}$

Notice that actually only the particular cases $k = 0$ and $k = 2$ of the branching constants $\wedge^1_k$ and $\vee^k_1$ are necessary. They are denoted by $\perp^1, \wedge^1$ and $\top_1, \vee_1$, respectively. In term of these constants the general branching constants are defined by

$$(*) \qquad \begin{array}{rcl} \wedge^1_0 & = & \perp^1 \\ \wedge^1_{k+1} & = & \wedge^1 \circ_{\mathrm{fd}} (\wedge^1_k +\!\!+ \mathsf{I}_1) \end{array} \qquad\qquad \begin{array}{rcl} \vee^0_1 & = & \top_1 \\ \vee^{k+1}_1 & = & (\vee^k_1 +\!\!+ \mathsf{I}_1) \circ_{\mathrm{fd}} \vee_1 \end{array}$$

These constants are extended to the block case, i.e., for $k, n \geq 0$ we introduce

$\mathsf{I}_n : n \to n, \quad {}^m\mathsf{X}^n : m + n \to n + m$

$\perp^n : n \to 0, \quad \wedge^n : n \to 2n, \quad \top_n : 0 \to n, \quad \vee_n : 2n \to n$

using as definitions the identities in axioms B6, B7–B8, A12–19 in Table 2. Finally, using the block extension of the definitions $(*)$ above we get meaning to the general block ramification and identification constants, i.e. for $k, n \geq 0$

$$\wedge_k^n : n \to kn, \qquad \vee_n^k : kn \to n$$

In this algebraic setting certain large but regular relations may be easily handled. For instance, short representations may be found, e.g.

$$(\vee_{343}^{524} \mathbin{+\!\!+} \wedge_{10^4}^{32}) \circ (\wedge_{50^5}^{43} \mathbin{+\!\!+} {}^{300}\mathsf{X}^1 \mathbin{+\!\!+} \vee_{101}^{99}) : 179764 \to 312500402$$

and operations may be computed in an easy way as in Section 6, but using general block constants.

Theorem 3.1 *The axioms without feedback in Table 2 and using A3^o/A7 version for the A3/A7 axioms give a correct and complete axiomatisation for relations with parallel and forward demonic sequential composition and $\mathsf{I}, \mathsf{X}, \perp, \wedge, \top, \vee$ constants. (No feedback is used here.)*

Proof: We follow the proof of the analogous result in the angelic case given in full detail and in a completely formal way in [Ste94]. Here we adapt the proof for the forward demonic case, but with a less formal presentation. An illustration of the formal computation proofs is given in Section 6.

I. First of all we note that the normal form representation of angelic relations used in [CaS91] may be used in this demonic case, as well. Hence, each relation $f : m \to n$ may be represented as

$$f = (\mathbin{+\!\!+}_{i \in [m]} \wedge_{m_i}^1) \circ_{\mathrm{fd}} b \circ_{\mathrm{fd}} (\mathbin{+\!\!+}_{j \in [n]} \vee_1^{n_i})$$

where $b : \Sigma_{i \in [m]} m_i \to \Sigma_{j \in [n]} n_j$ is a bijection and $m_i, n_j \geq 0$.

II. It is clear that each constant may be written in such a normal form. The normal form representation of such an f is not unique due to two reasons: (i) certain permutations may be done in an identification $(\vee_1^{n_i})$ or ramification $(\wedge_{m_i}^1)$ point and (ii) more that one path may connect an input-output pair. The proof that two normal form expressions that represent the same relation are equivalent in the specified axiomatic subsystem of Table 2 is the same as in the angelic case.

III. It is obvious that the parallel composite of two normal forms is a normal form, too. Hence we still have to show that the forward demonic sequential composite of two normal forms may be reduced to a normal form and, moreover, to show that this reduction may be done inside the axiomatic system presented in Table 2. We apply the same technique as in [CaS91, Ste94].

Following the indicated proofs, first of all we single out certain properties IP0–IP4 which follows from and generalize axioms A1–A11. They are presented in Table 1. There $\phi_{n,m} : m \times n \to n \times m$ is a bijection that naturally rearranges m groups of n elements in n groups of m elements. As one can see, only property IP1(b)$'$ is different from the angelic case. This difference produces some complications in the proof, as we shall see.

IP0	$\wedge_1^1 = \vee_1^1 = \mathsf{I}_1$
IP1(a)	$\wedge_m^1 \circ_{\mathrm{fd}} (\,\texttt{++}\,_{i\in[m]}\wedge_{m_i}^1) = \wedge^1_{\Sigma_{i\in[m]}m_i}$
IP1(b)$'$	$(\,\texttt{++}\,_{i\in[n]}\vee_1^{n_i}) \circ_{\mathrm{fd}} \vee_1^n = \begin{cases} \vee_1^{\Sigma_{i\in[n]}n_i} & \text{if } \forall i\in[n].n_i>0, \\ \perp^{\Sigma_{i\in[n]}n_i}\circ_{\mathrm{fd}} \top_1 & \text{if } \exists i\in[n].n_i=0 \end{cases}$
IP2(a)	$\wedge_m^1 \circ_{\mathrm{fd}} b = \wedge_m^1$, for all bijections $b: m\to m$
IP2(b)	$b \circ_{\mathrm{fd}} \vee_1^n = \vee_1^n$, for all bijections $b: n\to n$
IP3	$\vee_1^n \circ_{\mathrm{fd}} \wedge_m^1 = (\,\texttt{++}\,_{i\in[n]}\wedge_m^1) \circ_{\mathrm{fd}} \phi_{n,m} \circ_{\mathrm{fd}} (\,\texttt{++}\,_{i\in[m]}\vee_1^n)$
IP4	$\wedge_n^1 \circ_{\mathrm{fd}} \vee_1^n = \mathsf{I}_1, \qquad \forall n>0$

Table 1: Properties IP0–IP4 in the forward demonic case

The next step is to apply the above properties in order to normalize the sequential composite of two normal forms, say

$$[(\,\texttt{++}\,_{i\in[m]}\wedge_{m_i}^1) \circ_{\mathrm{fd}} b \circ_{\mathrm{fd}} (\,\texttt{++}\,_{j\in[n]}\vee_1^{n_i})] \circ_{\mathrm{fd}} [(\,\texttt{++}\,_{i'\in[n]}\wedge_{m'_{i'}}^1) \circ_{\mathrm{fd}} b' \circ_{\mathrm{fd}} (\,\texttt{++}\,_{j'\in[p]}\vee_1^{n'_{j'}})]$$

We start applying IP3 from left to right in order to commute the middle factors, i.e., the ramifications with indices $m'_{i'}$ pass over the identifications with indices n_i. Then,

(*) using general results in a symmetric strict monoidal category, these new ramifications produced by applying IP3 pass over the bijection b, as well, and finally with IP1(a) are eated by the ramifications with indices m_i.

A similar way take the identifications with indices n_i which arrive in front of the identifications with indices $n'_{j'}$. Till now it was similar to the angelic case. But in this point, when we apply IP1(b)$'$ the second case may occur. So, in general we get an expression of the following type

$$(\,\texttt{++}\,_{i\in[m]}\wedge_{..}^1) \circ_{\mathrm{fd}} b'' \circ_{\mathrm{fd}} (\mathsf{I}_{..} \,\texttt{++}\, \perp^k) \circ_{\mathrm{fd}} b''' \circ_{\mathrm{fd}} (\,\texttt{++}\,_{j'\in[p]}\vee_1^{..})$$

(In order to ease the writting we have collected all the resulting $\perp$'s in a unique summand by using appropriate bijections; in the same time, certain indices are not given in an explicit way, because their values are not relevant.) Hence certain additional 0-ramifications $\perp$'s occur, but fortunately they are eliminated applying the step (*) above and we get a normal form. □

The next result is an easy consequence.

Corollary 3.2 *Let $Y \subseteq \{\perp, \wedge, \top, \vee\}$ and denote by Y-REL^{fd} the class of the relations in REL^{fd} that are represented by terms written with constants in Y (and $\mathsf{I}, \mathsf{X}, \texttt{++}, \circ_{\mathrm{fd}}$). Then:*

– for $Y = \{\top, \vee\}$ (functions) or $Y = \{\wedge, \top, \vee\}$ (total relations) the class Y-REL^{fd} is not closed under forward demonic feedback;

– for the other subsets Y the classes Y-REL^{fd} are closed under the operations and the corresponding subsets of axioms in Theorem 3.1 axiomatize them.

4 Cyclic forward demonic case

In order to apply the axiomatisation we have given in the previous section in a cyclic context, one has to introduce a looping operator. We use the feedback $\uparrow$ operator introduced in [Ste90].

First we recall the definition of this usual (say "angelic") feedback $\uparrow$:
if $f : m+1 \to n+1$, then $f\uparrow^1 : m \to n$ is defined by

$$f\uparrow^1 = \{(i,j) : (i,j) \in f \text{ or } [(i,n+1) \in f \ \&\ (m+1,j) \in f]\}$$

(Since $f\uparrow^1 : m \to n$, it is clear that above $i \in [m], j \in [n]$.)

Next we complete our basic set of forward demonic operators with

- **forward demonic feedback** $\uparrow_{\mathrm{fd}}$:
  if $f : m+1 \to n+1$, then $f\uparrow^1_{\mathrm{fd}} : m \to n$ is defined by:

  – if $(m+1, n+1) \notin f$, then

$$f\uparrow^1_{\mathrm{fd}} = \{\ (i,j) :\ \begin{array}{l}(i,j) \in f \ \text{ and } \ [(m+1,j) \in f \Rightarrow \exists i' \in [m].(i', n+1) \in f] \\ \text{or} \\ (i,n+1) \in f \ \text{ and } \ (m+1,j) \in f \quad \}\end{array}$$

  – if $(m+1, n+1) \in f$, then

$$f\uparrow^1_{\mathrm{fd}} = \{\ (i,j) :\ (i,j) \in f \ \text{ and } \ (m+1,j) \notin f \quad \}$$

The definition is a bit boring. However, it may be easily seen that the denomic feedback $f\uparrow^1_{\mathrm{fd}}$ is included in the usual one $f\uparrow^1$. A more intuitive definition for $f\uparrow^1_{\mathrm{fd}}$ may be given using the graph $G(f\uparrow^1_{\mathrm{fd}})$ as follows:

> $(i,j) \in f\uparrow^1_{\mathrm{fd}}$ iff (1) $(i,j) \in f\uparrow^1$ and (2) every maximal path in $G(f\uparrow^1)$ ending in out_j is finite and it starts from an in_k with $k \in [m]$.

The second condition states that one connection $(i,j) \in f\uparrow^1$ is *not* a connection in $f\uparrow^1_{\mathrm{fd}}$ in two cases:

- $(m+1,j) \in f$ and $(m+1,n+1) \in f$; notice that in such a case there exists an infinite path ending in out_j;

- $(m+1,j) \in f$ and $\nexists(k,n+1) \in f$; in such a case $out_{n+1} \to in_{m+1} \to out_j$ is a maximal path which fails to start from an input vertex in_k, with $k \in [m]$.

Hence the direct definition and the one in terms of graphs for the forward demonic feedback are equivalent.

Theorem 4.1 *The axioms in Table 2 with the A3°/A7/F5° version for the A3/A7/F5 axioms give a correct and complete axiomatisation for relations with forward demonic operators.*

Proof: The proof is based on the similar result in the acyclic case (Theorem 3.1). The remaining proof may be obtained from the proof of the corresponding result in the angelic case (i.e., Theorem 6.1.4 in [Ste94]) using axiom F5° instead of F5 in the specific case (b).ii on page 83 in the cited paper. □

The following corollary may now be easily obtained.

Corollary 4.2 *Seven subclasses of relations Y-REL^{fd} with $Y \subseteq \{\perp, \wedge, \top, \vee\}$ are not closed under the forward demonic operators, i.e. for $Y = \{\vee\}$ (surjective functions), $Y = \{\top, \vee\}$ (functions), $Y = \{\wedge\}$ (converses of surjective functions), $Y = \{\wedge, \perp\}$ (converses of functions), $Y = \{\wedge, \top, \vee\}$ (total and surjective relations), $Y = \{\wedge, \top, \vee\}$ (total relations), and $Y = \{\wedge, \perp, \vee\}$ (surjective relations).*

The remaining nine classes are closed under the forward demonic operators and the corresponding subsets of axioms in the above theorm give axiomatisations for them. □

5 Two-ways demonic case

In the previous sections we have studied relations with forward demonic operators, namely the case when

(a) a maximal i-disconnected incoming path to an output vertex v destroyes the access of other incoming paths to v.

One may consider the *backward demonic* case, as well, i.e. the case given by the dual condition

(b) a maximal o-disconnected outgoing path from an input vertex s destroys the creation of other outgoing paths from s.

Using duality, it follows that one can axiomatize the backward demonic calculus of relations by the axioms in Table 2 in which the A3/A7°/F5° version for the A3/A7/F5 axioms is used.

Finally, one may consider the *two–way demonic* case in which both conditions (a) and (b) are required. The main result of this section is the following theorem.

Theorem 5.1 *An axiomatisation for relations with two–way demonic operators is given by the axioms in Table 2, where the A3°/A7°/F5° version for the A3/A7/F5 axioms is used.*

Proof: The proof of the fact that a two-way demonic sequential composite $f \circ_{\mathrm{fbd}} g$ of two normal forms may be reduced to a normal form using properties IP0, IP1(a)′, IP1(b)′, IP2–IP4 is now a bit more complicate.[3]

[3] IP1(a)′ is the dual of IP1(b)′.

1. We start with the same procedure as in the forward demonic case moving the middle ramifications and identifications in $E_0 := f \circ_{\text{fbd}} g$ near the similar ones which occur in the extremal factors, but now applying IP1(a)′ and IP1(b)′. Hence additional terms $\top$'s and $\perp$'s may occur. Let E_1 be the resulting term.

2. Repet 1, starting with E_1 and obtaining E_2. And so on.

One may see that at each application of 1 if additional terms occur, then at least one path/edge is deleted from the corresponding graph. Hence, the algorithm stops in a finite number of steps and we get a normal form for $f \circ_{\text{fbd}} g$.

The rest of the proof is similar to the corresponding proof in the forward demonic case. □

6 Simple algebraic computations

Here we present some simple computations which justifyes in an algebraic way the results presented in Example 2.1. Along the way computational insights for the proofs of the theorems in this paper may be obtained.

We start with the relational term presented before

$$f = (\wedge^1 \mathbin{+\!\!+} \wedge^1 \mathbin{+\!\!+} \mathsf{I}_1 \mathbin{+\!\!+} \perp^1) \circ (\mathsf{I}_1 \mathbin{+\!\!+} \vee_1 \circ(\top_1 \mathbin{+\!\!+} \mathsf{I}_1) \circ \vee_1 \mathbin{+\!\!+} \vee_1 \circ[(\wedge^2 \circ (\mathsf{I}_2 \mathbin{+\!\!+} \vee_1)) \uparrow^1])$$

and compute the normal form in various relational calculi. In order to have an uniform notation in this section we denote the angelic operators by $\circ_{\text{a}}$ and $\uparrow_{\text{a}}$.

Angelic case: First we note that

$$\begin{aligned}
&(\wedge^2 \circ_{\text{a}} (\mathsf{I}_2 \mathbin{+\!\!+} \vee_1)) \uparrow^1_{\text{a}} \\
&\quad = [(\wedge^1 \mathbin{+\!\!+} \wedge^1) \circ_{\text{a}} (\mathsf{I}_1 \mathbin{+\!\!+} {}^1\mathsf{X}^1 \mathbin{+\!\!+} \mathsf{I}_1) \circ_{\text{a}} (\mathsf{I}_2 \mathbin{+\!\!+} \vee_1)] \uparrow^1_{\text{a}} && \text{A19}\\
&\quad = \wedge^1 \circ_{\text{a}} ([(\mathsf{I}_2 \mathbin{+\!\!+} \wedge^1) \circ_{\text{a}} (\mathsf{I}_1 \mathbin{+\!\!+} {}^1\mathsf{X}^1 \mathbin{+\!\!+} \mathsf{I}_1) \circ_{\text{a}} (\mathsf{I}_2 \mathbin{+\!\!+} \vee_1)] \uparrow^1_{\text{a}}) && \text{R1}\\
&\quad = \wedge^1 \circ_{\text{a}} (\mathsf{I}_1 \mathbin{+\!\!+} [(\mathsf{I}_1 \mathbin{+\!\!+} \wedge^1) \circ_{\text{a}} ({}^1\mathsf{X}^1 \mathbin{+\!\!+} \mathsf{I}_1) \circ_{\text{a}} (\mathsf{I}_1 \mathbin{+\!\!+} \vee_1)] \uparrow^1_{\text{a}}) && \text{B5,R2}\\
&\quad = \wedge^1 \circ_{\text{a}} (\mathsf{I}_1 \mathbin{+\!\!+} \mathsf{I}_1) && \text{F5}\\
&\quad = \wedge^1
\end{aligned}$$

Notice that A3 and A7 are valid in this case. Hence actually we have a kind of sequential composite of two normal forms. Then it is normalized in the usual way,

$$\begin{aligned}
f \;&=\; (\wedge^1 \mathbin{+\!\!+} \wedge^1 \mathbin{+\!\!+} \mathsf{I}_1 \mathbin{+\!\!+} \perp^1) \circ_{\text{a}} (\mathsf{I}_1 \mathbin{+\!\!+} \vee_1 \circ_{\text{a}}(\top_1 \mathbin{+\!\!+} \mathsf{I}_1) \circ_{\text{a}} \vee_1 \mathbin{+\!\!+} \vee_1) \circ_{\text{a}} \wedge^1) \\
&=\; (\wedge^1 \mathbin{+\!\!+} \wedge^1 \mathbin{+\!\!+} \mathsf{I}_1 \mathbin{+\!\!+} \perp^1) \circ_{\text{a}} (\mathsf{I}_1 \mathbin{+\!\!+} \vee_1 \circ_{\text{a}} \mathsf{I}_1 \mathbin{+\!\!+} \vee_1 \circ_{\text{a}} \wedge^1) && \text{A3}\\
&=\; (\wedge^1 \mathbin{+\!\!+} \wedge^1 \mathbin{+\!\!+} \mathsf{I}_1 \mathbin{+\!\!+} \perp^1) \\
&\qquad \circ_{\text{a}} (\mathsf{I}_1 \mathbin{+\!\!+} \vee_1 \mathbin{+\!\!+} (\wedge^1 \mathbin{+\!\!+} \wedge^1) \circ_{\text{a}} (\mathsf{I}_1 \mathbin{+\!\!+} {}^1\mathsf{X}^1 \mathbin{+\!\!+} \mathsf{I}_1) \circ_{\text{a}} (\vee_1 \mathbin{+\!\!+} \vee_1)) && \text{IP3}\\
&=\; (\wedge^1 \mathbin{+\!\!+} \wedge^1 \mathbin{+\!\!+} \mathsf{I}_1 \mathbin{+\!\!+} \perp^1) \circ_{\text{a}} (\mathsf{I}_1 \mathbin{+\!\!+} \mathsf{I}_2 \mathbin{+\!\!+} \wedge^1 \mathbin{+\!\!+} \wedge^1) \\
&\qquad \circ_{\text{a}} (\mathsf{I}_4 \mathbin{+\!\!+} {}^1\mathsf{X}^1 \mathbin{+\!\!+} \mathsf{I}_1) \circ_{\text{a}} (\mathsf{I}_1 \mathbin{+\!\!+} \vee_1 \mathbin{+\!\!+} \vee_1 \mathbin{+\!\!+} \vee_1) \\
&=\; (\wedge^1 \mathbin{+\!\!+} \wedge^1 \circ_{\text{a}}(\mathsf{I}_1 \mathbin{+\!\!+} \wedge^1) \mathbin{+\!\!+} \wedge^1 \mathbin{+\!\!+} \perp^1) \\
&\qquad \circ_{\text{a}} (\mathsf{I}_4 \mathbin{+\!\!+} {}^1\mathsf{X}^1 \mathbin{+\!\!+} \mathsf{I}_1) \circ_{\text{a}} (\mathsf{I}_1 \mathbin{+\!\!+} \vee_1 \mathbin{+\!\!+} \vee_1 \mathbin{+\!\!+} \vee_1) \\
&=\; (\wedge^1 \mathbin{+\!\!+} \wedge^1_3 \mathbin{+\!\!+} \wedge^1 \mathbin{+\!\!+} \perp^1) \circ_{\text{a}} (\mathsf{I}_4 \mathbin{+\!\!+} {}^1\mathsf{X}^1 \mathbin{+\!\!+} \mathsf{I}_1) \circ_{\text{a}} (\mathsf{I}_1 \mathbin{+\!\!+} \vee_1 \mathbin{+\!\!+} \vee_1 \mathbin{+\!\!+} \vee_1)
\end{aligned}$$

What we have got is just the normal form of the resulting relation computed in the angelic case.

Forward demonic case: As in the previous case we have

$$\begin{aligned}
&(\wedge^2 \circ_{\mathrm{fd}} (\mathsf{I}_2 +\!\!\!+ \vee_1)) \uparrow^1_{\mathrm{fd}} \\
&\quad \ldots \\
&\quad = \wedge^1 \circ_{\mathrm{fd}} (\mathsf{I}_1 +\!\!\!+ [(\mathsf{I}_1 +\!\!\!+ \wedge^1) \circ_{\mathrm{fd}} ({}^1\mathsf{X}^1 +\!\!\!+ \mathsf{I}_1) \circ_{\mathrm{fd}} (\mathsf{I}_1 +\!\!\!+ \vee_1)] \uparrow^1_{\mathrm{fd}}) \\
&\quad = \wedge^1 \circ_{\mathrm{fd}} (\mathsf{I}_1 +\!\!\!+ \perp^1 \circ_{\mathrm{fd}} \top_1) && \mathrm{F5}^o
\end{aligned}$$

Then, using $\mathrm{A3}^o$ and A7 we get

$$\begin{aligned}
f &= (\wedge^1 +\!\!\!+ \wedge^1 +\!\!\!+ \mathsf{I}_1 +\!\!\!+ \perp^1) \\
&\qquad \circ_{\mathrm{fd}} (\mathsf{I}_1 +\!\!\!+ \vee_1 \circ_{\mathrm{fd}} (\top_1 +\!\!\!+ \mathsf{I}_1) \circ_{\mathrm{fd}} \vee_1 +\!\!\!+ \vee_1 \circ_{\mathrm{fd}} [\wedge^1 \circ_{\mathrm{fd}} (\mathsf{I}_1 +\!\!\!+ \perp^1 \circ_{\mathrm{fd}} \top_1)]) \\
&= (\wedge^1 +\!\!\!+ \wedge^1 +\!\!\!+ \mathsf{I}_1 +\!\!\!+ \perp^1) \\
&\qquad \circ_{\mathrm{fd}} (\mathsf{I}_1 +\!\!\!+ \vee_1 \circ_{\mathrm{fd}} \perp^1 \circ_{\mathrm{fd}} \top_1 +\!\!\!+ \vee_1 \circ_{\mathrm{fd}} \wedge^1 \circ_{\mathrm{fd}} (\mathsf{I}_1 +\!\!\!+ \perp^1) +\!\!\!+ \top_1) && \mathrm{A3}^o \\
&= (\wedge^1 +\!\!\!+ \wedge^1 +\!\!\!+ \mathsf{I}_1 +\!\!\!+ \perp^1) \\
&\qquad \circ_{\mathrm{fd}} (\mathsf{I}_1 +\!\!\!+ \perp^2 +\!\!\!+ \mathsf{I}_2) \circ_{\mathrm{fd}} (\mathsf{I}_1 +\!\!\!+ \top_1 +\!\!\!+ \vee_1 \circ_{\mathrm{fd}} \mathsf{I}_1 +\!\!\!+ \top_1)) && \mathrm{A7} \\
&= (\wedge^1 \circ_{\mathrm{fd}} (\mathsf{I}_1 +\!\!\!+ \top_1) +\!\!\!+ \wedge^1 \circ_{\mathrm{fd}} (\perp^1 +\!\!\!+ \mathsf{I}_1) +\!\!\!+ \mathsf{I}_1 +\!\!\!+ \perp^1) \circ_{\mathrm{fd}} (\mathsf{I}_1 +\!\!\!+ \top_1 +\!\!\!+ \vee_1 +\!\!\!+ \top_1) \\
&= (\mathsf{I}_1 +\!\!\!+ \mathsf{I}_1 +\!\!\!+ \mathsf{I}_1 +\!\!\!+ \perp^1) \circ_{\mathrm{fd}} (\mathsf{I}_1 +\!\!\!+ \top_1 +\!\!\!+ \vee_1 +\!\!\!+ \top_1) && \mathrm{A7}
\end{aligned}$$

which is the normal form of the resulting relation computed in the forward demonic case.

Backward demonic case: As in the previous forward demonic case we have

$$(\wedge^2 \circ_{\mathrm{bd}} (\mathsf{I}_2 +\!\!\!+ \vee_1)) \uparrow^1_{\mathrm{bd}} = \wedge^1 \circ_{\mathrm{bd}} (\mathsf{I}_1 +\!\!\!+ \perp^1 \circ_{\mathrm{bd}} \top_1)$$

Then, using A3 and $\mathrm{A7}^o$ we get

$$\begin{aligned}
f &= (\wedge^1 +\!\!\!+ \wedge^1 +\!\!\!+ \mathsf{I}_1 +\!\!\!+ \perp^1) \\
&\qquad \circ_{\mathrm{bd}} (\mathsf{I}_1 +\!\!\!+ \vee_1 \circ_{\mathrm{bd}} (\top_1 +\!\!\!+ \vee_1) \circ_{\mathrm{bd}} \vee_1 +\!\!\!+ \vee_1 \circ_{\mathrm{bd}} [\wedge^1 \circ_{\mathrm{bd}} (\mathsf{I}_1 +\!\!\!+ \perp^1 \circ_{\mathrm{bd}} \top_1)]) \\
&= (\wedge^1 +\!\!\!+ \wedge^1 +\!\!\!+ \mathsf{I}_1 +\!\!\!+ \perp^1) \\
&\qquad \circ_{\mathrm{bd}} (\mathsf{I}_1 +\!\!\!+ \vee_1 \circ_{\mathrm{bd}} \mathsf{I}_1 +\!\!\!+ \vee_1 \circ_{\mathrm{bd}} \wedge^1 \circ_{\mathrm{bd}} (\mathsf{I}_1 +\!\!\!+ \perp^1) +\!\!\!+ \top_1) && \mathrm{A3} \\
&= (\wedge^1 +\!\!\!+ \wedge^1 +\!\!\!+ \mathsf{I}_1 +\!\!\!+ \perp^1) \\
&\qquad \circ_{\mathrm{bd}} (\mathsf{I}_1 +\!\!\!+ \vee_1 +\!\!\!+ \vee_1 \circ_{\mathrm{bd}} \perp^1 \circ_{\mathrm{bd}} \top_1 +\!\!\!+ \top_1) && \mathrm{A7}^o \\
&= (\wedge^1 +\!\!\!+ \wedge^1 +\!\!\!+ \mathsf{I}_1 +\!\!\!+ \perp^1) \circ_{\mathrm{bd}} (\mathsf{I}_1 +\!\!\!+ \vee_1 +\!\!\!+ \perp^2 +\!\!\!+ \top_1 +\!\!\!+ \top_1) \\
&= (\wedge^1 +\!\!\!+ \wedge^1 \circ_{\mathrm{bd}} (\mathsf{I}_1 +\!\!\!+ \perp^1) +\!\!\!+ \perp^1 +\!\!\!+ \perp^1) \circ_{\mathrm{bd}} (\mathsf{I}_1 +\!\!\!+ \vee_1 +\!\!\!+ \top_2) \\
&= (\wedge^1 +\!\!\!+ \perp^1 \circ_{\mathrm{bd}} \top_1 +\!\!\!+ \perp^2) \circ_{\mathrm{bd}} (\mathsf{I}_1 +\!\!\!+ \vee_1 +\!\!\!+ \top_2) && \mathrm{A7}^o \\
&= (\wedge^1 +\!\!\!+ \perp^1 +\!\!\!+ \perp^2) \circ_{\mathrm{bd}} (\mathsf{I}_1 +\!\!\!+ (\mathsf{I}_1 +\!\!\!+ \top_1) \circ_{\mathrm{bd}} \vee_1 +\!\!\!+ \top_2) \\
&= (\wedge^1 +\!\!\!+ \perp^3) \circ_{\mathrm{bd}} (\mathsf{I}_1 +\!\!\!+ \mathsf{I}_1 +\!\!\!+ \top_2) && \mathrm{A3}
\end{aligned}$$

which is the normal form of the expected result computed in the backward demonic case.

Two-way demonic case: The feedback part gives the same result as in previous demonic cases. Using $\mathrm{A3}^o$ and $\mathrm{A7}^o$ we may compute

$$
\begin{aligned}
f &= (\wedge^1 \mathbin{+\!\!+} \wedge^1 \mathbin{+\!\!+} \mathsf{I}_1 \mathbin{+\!\!+} \perp^1) \\
&\qquad \circ_{\mathrm{fbd}} (\mathsf{I}_1 \mathbin{+\!\!+} \vee_1 \circ_{\mathrm{fbd}} (\top_1 \mathbin{+\!\!+} \mathsf{I}_1) \circ_{\mathrm{fbd}} \vee_1 \mathbin{+\!\!+} \vee_1 \circ_{\mathrm{fbd}} [\wedge^1 \circ_{\mathrm{fbd}} (\mathsf{I}_1 \mathbin{+\!\!+} \perp^1 \circ_{\mathrm{fbd}} \top_1)]) \\
&= (\wedge^1 \mathbin{+\!\!+} \wedge^1 \mathbin{+\!\!+} \mathsf{I}_1 \mathbin{+\!\!+} \perp^1) \\
&\qquad \circ_{\mathrm{fbd}} (\mathsf{I}_1 \mathbin{+\!\!+} \vee_1 \circ_{\mathrm{fbd}} \perp^1 \circ_{\mathrm{fbd}} \top_1 \mathbin{+\!\!+} \vee_1 \circ_{\mathrm{fbd}} [\wedge^1 \circ_{\mathrm{fbd}} (\mathsf{I}_1 \mathbin{+\!\!+} \perp^1)] \mathbin{+\!\!+} \top_1) && \mathrm{A3}^o \\
&= (\wedge^1 \mathbin{+\!\!+} \wedge^1 \mathbin{+\!\!+} \mathsf{I}_1 \mathbin{+\!\!+} \perp^1) \\
&\qquad \circ_{\mathrm{fbd}} (\mathsf{I}_1 \mathbin{+\!\!+} \perp^2 \mathbin{+\!\!+} \top_1 \mathbin{+\!\!+} \vee_1 \circ_{\mathrm{fbd}} \perp^1 \circ_{\mathrm{fbd}} \top_1 \mathbin{+\!\!+} \top_1) && \mathrm{A7}^o \\
&= (\wedge^1 \mathbin{+\!\!+} \wedge^1 \mathbin{+\!\!+} \mathsf{I}_1 \mathbin{+\!\!+} \perp^1) \circ_{\mathrm{fbd}} (\mathsf{I}_1 \mathbin{+\!\!+} \perp^2 \mathbin{+\!\!+} \top_1 \mathbin{+\!\!+} \perp^2 \mathbin{+\!\!+} \top_1 \mathbin{+\!\!+} \top_1) \\
&= (\wedge^1 \circ_{\mathrm{fbd}} (\mathsf{I}_1 \mathbin{+\!\!+} \perp^1) \mathbin{+\!\!+} \wedge^1 \circ_{\mathrm{fbd}} (\perp^1 \mathbin{+\!\!+} \perp^1) \mathbin{+\!\!+} \perp^1 \mathbin{+\!\!+} \perp^1) \\
&\qquad \circ_{\mathrm{fbd}} (\mathsf{I}_1 \mathbin{+\!\!+} \top_1 \mathbin{+\!\!+} \top_2) && \mathrm{A7}^o \\
&= (\perp^1 \circ_{\mathrm{fbd}} \top_1 \mathbin{+\!\!+} \perp^1 \mathbin{+\!\!+} \perp^2) \circ_{\mathrm{fbd}} (\mathsf{I}_1 \mathbin{+\!\!+} \top_3) \\
&= (\perp^1 \mathbin{+\!\!+} \perp^3) \circ_{\mathrm{fbd}} (\top_1 \mathbin{+\!\!+} \top_3) \\
&= \perp^4 \circ_{\mathrm{fbd}} \top_4
\end{aligned}
$$

which is the normal form of the expected result computed in the two-way demonic case.

7 Closing remarks

Synchronous dataflow networks: We have provided network algebras for classes of relations with forward demonic operators. This result may be combined with some general results to get network algebra models for SDFNs. More precisely, if one adds a certain (doubly-ranked) set X of proper cells to T, then using e.g. Theorem 5.4.2 in [Ste94], it folows that the resulting SDFNs over X and T modulo isomorphism form the free network algebra generated by X and T. In addition, this allow to use initial algebra semantics for SDFNs.

Many-sorted relations: In the previous sections, for the sake of simplicity we have studied the homogenous case. However, the results hold for the heterogenous case, as well. This means, various channel types may be used to model the input or output interfaces.

We give some details. Let S be a set of channel types and $a, b \in S^*$. A string $a \in S^*$ is written as $a = a_1 + \ldots + a_{|a|}$, where a_i denotes the i-th letter of a. Then an *S-sorted relation* $f \in \mathsf{REL}_S(a, b)$ is a usual relation which preserve the channel type, i.e.

$$\mathsf{REL}_S(a, b) = \{f \subseteq [|a|] \times [|b|] : (i, j) \in f \Rightarrow a_i = b_j\}$$

The previous case is just the particular instance of this more general case where S has exactly one element.

In this many-sorted framework we may usefully present the correctness and completeness result as an initial algebra theorem which allows a simple way to define the refinement of relations. Namely, for each enriched ssmc with feedback R satisfying the underlying axioms and each function $\phi : S \to Ob(R)$ from S to the objects of R there exists a unique extension of ϕ to a morphism $\phi^\sharp : \mathsf{REL}_S \to R$. In particular, if one refines some channels in S by elements in X^*, say, then this refinement is automatically extended to all S-sorted relations.

Complexity of the representation(s) of relations: In the calculi of relations we have presented so far short representations for certain large, but regular relations may be given, e.g. $(\vee^{524}_{343} \mathbin{+\!\!+} \wedge^{32}_{10^4}) \circ (\wedge^{43}_{5^5} \mathbin{+\!\!+} {}^{300}\mathsf{X}^1 \mathbin{+\!\!+} \vee^{99}_{101})$. However, this is not the case for all relations. An open question is whether better algebraic representations may be found.

To conclude, we have presented network algebra axiomatisations for relations with various demonic (including looping) operators. One particular case, namely that of relations with forward demonic operators, was used to model connections in synchronous dataflow networks. It turned out to be useful to present relations as enriched symmetric strict monoidal categories with feedback. Some slight changes of two or three axioms in the axiomatisation for relations with angelic operators in [CaS91, Ste94] are sufficient to model relations with three types of demonic operators.

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8 Appendix: The axioms

Table 2: Network algebra axioms for relations with angelic/demonic operators; $\circ/\dagger$ denotes various angelic-demonic versions for sequential/feedback operators; from each pair (A3,A3^o), (A7,A7^o) and (F5,F5^o) one axiom is chosen in each particular case.

B1	$f \mathbin{+\!\!+} (g \mathbin{+\!\!+} h) = (f \mathbin{+\!\!+} g) \mathbin{+\!\!+} h$	B6	$\mathsf{I}_a \mathbin{+\!\!+} \mathsf{I}_b = \mathsf{I}_{a+b}$
B2	$\mathsf{I}_0 \mathbin{+\!\!+} f = f = f \mathbin{+\!\!+} \mathsf{I}_0$	B7	${}^a\mathsf{X}^b \circ {}^b\mathsf{X}^a = \mathsf{I}_{a+b}$
B3	$f \circ (g \circ h) = (f \circ g) \circ h$	B8	${}^a\mathsf{X}^0 = \mathsf{I}_a$
B4	$\mathsf{I}_a \circ f = f = f \circ \mathsf{I}_b$	B9	${}^a\mathsf{X}^{b+c} = ({}^a\mathsf{X}^b \mathbin{+\!\!+} \mathsf{I}_c) \circ (\mathsf{I}_b \mathbin{+\!\!+} {}^a\mathsf{X}^c)$
B5	$(f \mathbin{+\!\!+} f') \circ (g \mathbin{+\!\!+} g') = f \circ g \mathbin{+\!\!+} f' \circ g'$	B10	$(f \mathbin{+\!\!+} g) \circ {}^c\mathsf{X}^d = {}^a\mathsf{X}^b \circ (g \mathbin{+\!\!+} f)$ for $f : a \to c, g : b \to d$

I. Axioms for ssmc-ies (symmetric strict monoidal categories)

A1	$(\vee_a \mathbin{+\!\!+} \mathsf{I}_a) \circ \vee_a = (\mathsf{I}_a \mathbin{+\!\!+} \vee_a) \circ \vee_a$	A5	$\wedge^a \circ (\wedge^a \mathbin{+\!\!+} \mathsf{I}_a) = \wedge^a \circ (\mathsf{I}_a \mathbin{+\!\!+} \wedge^a)$
A2	${}^a\mathsf{X}^a \circ \vee_a = \vee_a$	A6	$\wedge^a \circ {}^a\mathsf{X}^a = \wedge^a$
A3	$(\top_a \mathbin{+\!\!+} \mathsf{I}_a) \circ \vee_a = \mathsf{I}_a$	A7	$\wedge^a \circ (\perp^a \mathbin{+\!\!+} \mathsf{I}_a) = \mathsf{I}_a$
A3^o(*)	$(\top_a \mathbin{+\!\!+} \mathsf{I}_a) \circ \vee_a = \perp^a \circ \top_a$	A7^o(*)	$\wedge^a \circ (\perp^a \mathbin{+\!\!+} \mathsf{I}_a) = \perp^a \circ \top_a$
A4	$\vee_a \circ \perp^a = \perp^a \mathbin{+\!\!+} \perp^a$	A8	$\top_a \circ \wedge^a = \top_a \mathbin{+\!\!+} \top_a$

A9 $\top_a \circ \perp^a = \mathsf{I}_0$
A10 $\vee_a \circ \wedge^a = (\wedge^a \mathbin{+\!\!+} \wedge^a) \circ (\mathsf{I}_a \mathbin{+\!\!+} {}^a\mathsf{X}^a \mathbin{+\!\!+} \mathsf{I}_a) \circ (\vee_a \mathbin{+\!\!+} \vee_a)$
A11 $\wedge^a \circ \vee_a = \mathsf{I}_a$

A12	$\top_0 = \mathsf{I}_0$	A16	$\perp^0 = \mathsf{I}_0$
A13	$\top_{a+b} = \top_a \mathbin{+\!\!+} \top_b$	A17	$\perp^{a+b} = \perp^a \mathbin{+\!\!+} \perp^b$
A14	$\vee_0 = \mathsf{I}_0$	A18	$\wedge^0 = \mathsf{I}_0$
A15	$\vee_{a+b} = (\mathsf{I}_a \mathbin{+\!\!+} {}^b\mathsf{X}^a \mathbin{+\!\!+} \mathsf{I}_b) \circ (\vee_a \mathbin{+\!\!+} \vee_b)$	A19	$\wedge^{a+b} = (\wedge^a \mathbin{+\!\!+} \wedge^b) \circ (\mathsf{I}_a \mathbin{+\!\!+} {}^a\mathsf{X}^b \mathbin{+\!\!+} \mathsf{I}_b)$

II. Axioms for the additional constants $\top, \perp, \vee, \wedge$ (without feedback)

R1	$f \circ (g\dagger^c) \circ h = ((f \mathbin{+\!\!+} \mathsf{I}_c) \circ g \circ (h \mathbin{+\!\!+} \mathsf{I}_c))\dagger^c$	(relating "$\dagger$" and "$\circ$")
R2	$f \mathbin{+\!\!+} g\dagger^c = (f \mathbin{+\!\!+} g)\dagger^c$	(relating "$\dagger$" and "$\mathbin{+\!\!+}$")
R3	$(f \circ (\mathsf{I}_b \mathbin{+\!\!+} g)) \uparrow^c = ((\mathsf{I}_a \mathbin{+\!\!+} g) \circ f) \uparrow^d$ for $f : a + c \to b + d, g : d \to c$	(shifting blocks on feedback)
R4	$f\dagger^0 = f$	(no feedback)
R5	$(f\dagger^b) \uparrow^a = f\dagger^{a+b}$	(multiple feedbacks)

III. Axioms for feedback

F1 $\mathsf{I}_a\dagger^a = \mathsf{I}_0$
F2 ${}^a\mathsf{X}^a\dagger^a = \mathsf{I}_a$
F3 $\vee_a\dagger^a = \perp^a$ F4 $\wedge^a\dagger^a = \top_a$
F5 $[(\mathsf{I}_a \mathbin{+\!\!+} \wedge^a) \circ ({}^a\mathsf{X}^a \mathbin{+\!\!+} \mathsf{I}_a) \circ (\mathsf{I}_a \mathbin{+\!\!+} \vee_a)]\dagger^a = \mathsf{I}_a$
F5^o(*) $[(\mathsf{I}_a \mathbin{+\!\!+} \wedge^a) \circ ({}^a\mathsf{X}^a \mathbin{+\!\!+} \mathsf{I}_a) \circ (\mathsf{I}_a \mathbin{+\!\!+} \vee_a)]\dagger^a = \perp^a \circ \top_a$

IV. Axioms for the action of feedback on constants
