MODULE ALGEBRA FOR INITIAL ALGEBRA SEMANTICS

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It is shown that specification modules may be interpreted by a free functor construction on arbitrary algebras in such a way that most of the laws of the module algebra of Bergstra, Heering and Klint are satisfied; in particular, the normal form theorem for closed module expressions holds. By way of conclusion, two explications of parameter passing are pointed out.

1. Introduction

In [BHK], Bergstra, Heering and Klint describe a simple calculus of symmetric combination and export (BMA, for *basic module algebra*), which they hope may serve as a foundation for a formal theory of modularization. They treat a number of interpretations of this calculus, concentrating in particular on model classes and first order theories. (The pertinent ideas were applied in work on COLD [FJ] and VDM [Mi].) With regard to initial algebra semantics, they only suggest a rather simple-minded derivative of the model class interpretation, which they immediately show to be untenable. Below a more sophisticated approach will be taken: a straightforward generalization of initial algebra semantics will be described that constitutes a model of the essentials of module algebra.

Some generalization appears to be unavoidable, since there is no composition of initial algebras to correspond to the simple combination of specifications.

1.1 Example (BERGSTRA). Define $\mathcal{S}_n$, for $n \in \mathbb{N}$, with signature

$$\{S,\ a{:}S,\ I{:}S \rightarrow S\}$$

by the axioms Ia=a and I^nx=x (I^n stands for n applications of I). All $\mathcal{S}_n$ have the same, trivial, initial algebra; but to an additional constant b:S they all react differently.

The intuitive argument for the actual generalization proposed below runs as follows. Suppose we have some equational specification $\mathcal{T}$; of natural numbers, say, or of sequences of items of some assumed sort. Then $\mathcal{T}$ adds data and operations to whatever it was that we already had. What we already had must have been some complex of data and operations — an algebra **A**; and adding the specified numbers, sequences or what have you, we get an algebra $F_{\mathcal{T}}(\mathbf{A})$. So at the heart of a module, we have a construction $F_{\mathcal{T}}$ of algebras out of algebras, similar to the ‘free construction’ that Thatcher, Wagner and Wright used as the basis of the theory of parametrized specifications [TWW]. However, the module is not just this construction: it is also the way in which $F_{\mathcal{T}}(\mathbf{A})$ is related to $\mathbf{A}$, in particular, which elements of $F_{\mathcal{T}}(\mathbf{A})$ represent which elements of $\mathbf{A}$. Now suppose that we expect the construction $F_{\mathcal{T}}$ to respect the fundamental algebraic relations between its inputs. Indeed, if we generalize the concept of homomorphism, so that a homomorphism in the wider sense from $\mathbf{A}$ to $\mathbf{B}$ is a homomorphism in the old sense from $\mathbf{A}$ to a reduct of $\mathbf{B}$, we get a category **Alg** of all algebras; then these wider homomorphisms will do for fundamental algebraic relations, and the decency expected of $F_{\mathcal{T}}$ is: that it is an endofunctor of **Alg**. Now it is natural to present the relation between $\mathbf{A}$ and $F_{\mathcal{T}}(\mathbf{A})$ as a homomorphism $\mathbf{A} \to F_{\mathcal{T}}(\mathbf{A})$; and uniformly for all algebras, as a natural transformation from the identity functor of **Alg** to $F_{\mathcal{T}}$. Then finally the module given by $\mathcal{T}$ is the isomorphism class — isomorphism in a sense to be specified below — of this natural transformation. Generalizing away from syntactic entities such as $\mathcal{T}$, we arrive at the uniformity types of §8.

The definition of $F_{\mathcal{T}}$ is such that if $\mathcal{T}$ has an initial model $\mathbf{I}$, we have $F_{\mathcal{T}}(\varnothing) \cong \mathbf{I}$, where $\varnothing$ is the unique algebra of empty signature. It will follow, if $\mathcal{T}$ and $\mathcal{T}'$ have initial models $\mathbf{I}$ and $\mathbf{I}'$, and $\mathcal{T}$ and $\mathcal{T}'$ represent the same uniformity type, that $\mathbf{I} \cong \mathbf{I}'$. The converse is definitely false, as it should be, by example 1.1. In §7 an embedding will be constructed of the modules corresponding to flat equational specifications into a large class of varieties.

An effort has been made to keep this article reasonably self-contained.[1] Thus, §§2-5 are, at least in part, of a preliminary nature. §2 constructs signatures and their morphisms, and demonstrates some properties that appear to be essential in the sequel. After this, it is not much work to define equational theories. §4 is an epitome of module algebra, adapted and reduced to equational format. §5 describes the category **Alg** and defines the functors $F_{\mathcal{T}}$ and natural transformations $\eta_{\mathcal{T}}$. The latter are generalized in §6, where a combination operation + is defined. In §7 we return to the category **Alg** to study the role of signature. The model of module algebra that is the *pièce de resistance* of the undertaking appears in §8. §§9-11 contain some final considerations. A summary of elementary category theory that did not fit into the line of the argument appears as an appendix.

[1]That is, I have some confidence that enough is said about what the reader is expected to know in advance for readers who know to remember that they do.

2. Signatures and renamings

Let N be a countably infinite set: the set of *names*. The set F(N) of *function declarations* over N is the set of finite sequences of length at least 2 of elements of N. I assume as a matter of course that $\mathrm{N} \cap \mathrm{F(N)} = \emptyset$; similar assumptions in the sequel may remain implicit.

The set D(N) of *declarations over* N is $\mathrm{N} \cup \mathrm{F(N)}$; if we identify sequences of length 1 with their single element, then D(N) is the set of nonempty sequences of elements of N. A *signature* over N is a finite set σ of declarations over N with the property

$$\langle \mathrm{n}_0,\ldots,\mathrm{n}_k\rangle \in \sigma \Rightarrow \forall i(0 < i \leq k \Rightarrow \mathrm{n}_i \in \sigma).$$

The set of signatures over N will be denoted by Σ(N).

If σ is a signature, then $\mathrm{S}\sigma := \sigma \cap \mathrm{N}$ and $\mathrm{F}\sigma := \sigma \cap \mathrm{F(N)}$.

Let X be a set or sequence; then $|X|$ is the number of its elements.

2.1 Definition. Let σ and τ be signatures. A *morphism* from σ to τ is a map $\phi : \sigma \to \tau$ such that (for $k, l \geq 0$)

$$(1) \qquad \phi(\mathrm{n}_0, \ldots, \mathrm{n}_k) = \langle \mathrm{m}_0, \ldots, \mathrm{m}_l\rangle \Rightarrow l = k \;\&\; \forall i(0 < i \leq k \Rightarrow \mathrm{m}_i = \phi(\mathrm{n}_i)).$$

A bijective operation ϕ on D(N) that satisfies (1) will be called a D-*bijection*; an involutive D-bijection is a *renaming*. Instead of $\phi(x)$ I usually write $\phi\cdot x$, or ϕx; and for a set X, $\phi\cdot X = \{\phi\cdot x | x \in X\}$. Clearly $\phi\cdot\sigma \in \Sigma(\mathrm{N})$ if $\sigma \in \Sigma(\mathrm{N})$. If ϕ is a D-bijection, then so is ϕ^{-1}.

I assume given an enumeration

$$\mathrm{n}_0,\ \mathrm{n}_1,\ \mathrm{n}_2,\ \ldots,\ \mathrm{n}_k,\ \ldots \qquad (k \in \mathbb{N})$$

of N. I write $\mathrm{m} < \mathrm{n}$ if there exist $k < k'$ such that $\mathrm{m} = \mathrm{n}_k$ and $\mathrm{n} = \mathrm{n}_{k'}$. I assume an extension of the ordering of N to a well-ordering of D(N) of the same type with the property that $\mathbf{d} < \mathbf{e}$ if $\mathbf{d}$ is a subsequence of $\mathbf{e}$.

If $R \subseteq X \times Y$, and $S \subseteq X$, then $R(S) = \{y \in Y | \exists s \in S. sRy\}$. I write $R\{x\}$ instead of $R(\{x\})$. For any binary relation R, $D(R) = \{x | \exists y. \langle x, y\rangle \in R\}$, and $C(R) = \{x | \exists y. \langle y, x\rangle \in R\}$. A relation is *single-valued* (or a partial function) if $\forall xyz(xRy \;\&\; xRz \Rightarrow y = z)$. From functions $f_1 : X \to Y_1, \ldots, f_n : X \to Y_n$, a function from X to $Y_1 \times \ldots \times Y_n$ is defined by: $(f_1, \ldots, f_n)(x) = \langle f_1(x), \ldots, f_n(x)\rangle$.

2.2 Lemma. Let σ and τ be signatures, and $\phi : \sigma \to \tau$ an injective morphism. Then ϕ can be extended to a D-bijection.

Proof. Let $\mathrm{D(N)} = \{\mathbf{d}_i | i \in \mathbb{N}\}$, with $\mathbf{d}_j < \mathbf{d}_i$ if $j < i$. We define a sequence

$$\phi_0 \subseteq \phi_1 \subseteq \cdots \subseteq \phi_i \subseteq \cdots \qquad (i \in \mathbb{N})$$

of finite single-valued relations on D(N), with $\mathbf{d}_i \in D(\phi_{i+1}) \cap C(\phi_{i+1})$, as follows. We take $\phi_0 = \phi$. Now suppose ϕ_j has been constructed, and $i = j+1$. We first construct an auxiliary relation ϕ'_j. If $\mathbf{d}_j \in D(\phi_j)$, then $\phi'_j = \phi_j$. Otherwise, if

$\mathbf{d}_j = \mathrm{n} \in \mathrm{N}$, and n' is the first element of N that does not belong to $C(\phi_j)$, then $\phi_j' = \phi_j \cup \{\langle \mathrm{n}, \mathrm{n}' \rangle\}$. Now suppose $\mathbf{d}_j = \langle \mathrm{n}_0, \mathrm{n}_1, \ldots, \mathrm{n}_k \rangle$, with $k > 0$. Then $\mathrm{n}_l \in D(\phi_j)$, $1 \le l \le k$, by the choice of the ordering on D(N). Let n be the first element of N for which

$$\langle \mathrm{n}, \phi_j(\mathrm{n}_1), \ldots, \phi_j(\mathrm{n}_k) \rangle =: \mathbf{d} \notin C(\phi_j):$$

then $\phi_j' = \phi_j \cup \{\langle \mathbf{d}_j, \mathbf{d} \rangle\}$. Now if $\mathbf{d}_j \in C(\phi_j')$, we set $\phi_i = \phi_j'$. Otherwise we reason as above, with C and D interchanged; for the n'' or $\mathbf{d}'$ found on this way we define $\phi_i = \phi_j' \cup \{\langle \mathrm{n}'', \mathrm{n} \rangle\}$ or $\phi_i = \phi_j' \cup \{\langle \mathbf{d}', \mathbf{d} \rangle\}$ respectively.

Let

$$\overline{\phi} = \bigcup_{i \in \mathbb{N}} \phi_i ;$$

then evidently $\overline{\phi}$ is a D-bijection. ¶

2.3 Lemma. Let σ be a signature, and $r : \sigma \to \sigma$ an involutive morphism. Then r can be extended to a renaming.

Proof. A variation on the proof of the previous lemma. Again let D(N) = $\{\mathbf{d}_i \mid i \in \mathbb{N}\}$. We define a sequence

$$r_0 \subseteq r_1 \subseteq \ldots \subseteq r_i \subseteq \ldots \qquad (i \in \mathbb{N})$$

of finite partial operations on D(N), with $\mathbf{d}_i \in D(r_{i+1})$ and $D(r_i) = C(r_i)$, as follows. We take $r_0 = r$. If $i = j+1$, and $\mathbf{d}_j \in D(r_j)$, then $r_i = r_j$. If $\mathbf{d}_j \notin D(r_j)$, and $\mathbf{d}_j = \mathrm{n} \in \mathrm{N}$, then $r_i = r_j \cup \{\langle \mathrm{n}, \mathrm{n} \rangle\}$. Otherwise $\mathbf{d}_i = \langle \mathrm{n}_0, \mathrm{n}_1, \ldots, \mathrm{n}_k \rangle$, with $k > 0$. Then $\mathrm{n}_i \in D(r_j)$, $1 \le i \le k$, since by our choice of ordering $\mathrm{n}_i < \mathbf{d}_j$. Let n be the least element of N for which $\langle \mathrm{n}, r_j(\mathrm{n}_1), \ldots, r_j(\mathrm{n}_k) \rangle =: \mathbf{d} \notin D(r_j)$: then

$$r_i = r_j \cup \{\langle \mathbf{d}_j, \mathbf{d} \rangle, \langle \mathbf{d}, \mathbf{d}_j \rangle\}.$$

Let

$$s = \bigcup_{i \in \mathbb{N}} r_i ;$$

then evidently s is a renaming. ¶

Observe that $\sigma \cup \tau \in \Sigma(\mathrm{N})$ when $\sigma, \tau \in \Sigma(\mathrm{N})$. The inverse R^{-1} of a relation R is defined as $\{\langle x, y \rangle \mid \langle y, x \rangle \in R\}$.

2.4 Lemma. Let σ and τ be signatures, and $r : \sigma \to \tau$ a bijective morphism that satisfies

(2) $\quad \forall \mathbf{d} \in \sigma \cap \tau .\, r(\mathbf{d}) = \mathbf{d}$.

Then r is contained in a renaming.

Proof. Put $s = r \cup r^{-1}$. By condition (2) s is an automorphism of $\sigma\cup\tau$; it also is an involution. Apply lemma 3. ¶

2.5 Lemma. Let σ, τ, υ and ϕ be signatures. Then there exist renamings r and s such that $r|\sigma = 1_\sigma$, $s|\tau = 1_\tau$, and $r\cdot\upsilon \cap s\cdot\phi \subseteq \sigma\cap\tau$.

Proof. We may assume that $\sigma\subseteq\upsilon$ and $\tau\subseteq\phi$ (replace υ by $\sigma\cup\upsilon$, ϕ by $\tau\cup\phi$). We construct r in two steps: first we define a morphism r_0 of υ, then we apply lemma 4 to obtain a renaming.

For $\mathbf{d}\in\sigma$ we define: $r_0(\mathbf{d})=\mathbf{d}$. Let l be the greatest i such that n_i occurs in $\upsilon\cup\tau$. For $\mathrm{n}_i \in \mathrm{S}\upsilon-\sigma$ we define: $r_0(\mathrm{n}_i) = \mathrm{n}_{l+1+i}$. For $\mathbf{d} = \langle \mathrm{n}_j, \mathrm{m}_1, \ldots, \mathrm{m}_k\rangle \in \mathrm{F}\upsilon\sigma$ we let $r_0(\mathbf{d}) = \langle \mathrm{n}_{l+1+j}, r_0(\mathrm{m}_1), \ldots, r_0(\mathrm{m}_k)\rangle$.

It is easy to see that $r_0(\upsilon)$ is a signature and r_0 an injective morphism (by the choice of l), and that $\forall\mathbf{d}\in \upsilon\cap r_0(\upsilon): r_0(\mathbf{d})=\mathbf{d}$. By lemma 4, r_0 is contained in a renaming r.

The construction of s is symmetric, except that the part of τ is not taken over by σ, but by $r\cdot\upsilon$. ¶

Let σ be a signature and r a renaming. We define σ / r as

$$\{\mathbf{d}\in\sigma \,|\, r\cdot\mathbf{d}=\mathbf{d}\};$$

σ / r is a signature.

3. Equational theories

Let σ be a signature. A *variable* of signature σ (or: *for* σ) is a pair $\mathbf{x} = \langle \mathrm{x}, \mathrm{S}\rangle$ of a symbol x that does not occur in σ and a sort symbol of σ. Terms are considered to be built from such variables and function declarations. Precisely: let X be a set of variables of signature σ. Then (i) if $\mathbf{x} = \langle \mathrm{x}, \mathrm{S}\rangle \in \mathrm{X}$, then $\mathbf{x}$ is a σ-term in X of sort S; (ii) if $\mathbf{f} = \langle \mathrm{f}, \mathrm{S}_1, \ldots, \mathrm{S}_n\rangle \in \sigma$ ($n \geq 1$), and $\mathbf{t}_i$, for $1 \leq i < n$, is a σ-term in X of sort S_i, then $\mathbf{f}\mathbf{t}_1\ldots\mathbf{t}_{n-1}$ is a σ-term in X of sort S_n; and something can only be a σ-term on the strength of a finite sequence of applications of (i) and (ii). I also speak of terms over σ in variables from X, etc. For readability, terms as in (ii) may be written $\mathbf{f}(\mathbf{t}_1, \ldots, \mathbf{t}_{n-1})$. The sort of $\mathbf{t}$ will sometimes be denoted by $\mathrm{S}_\mathbf{t}$.

A σ-*equation* in X is a pair $\langle \mathbf{s}, \mathbf{t}\rangle$ of σ-terms in X of the same sort. Of course I always write $\mathbf{s}=\mathbf{t}$.

An *equational theory* is a pair $\mathcal{A} = \langle\sigma, \mathrm{A}\rangle$ of a signature σ and a set A of equations over σ in some set X of variables of signature σ; the *signature* $\Sigma\mathcal{A}$ of $\mathcal{A}$ is σ.[1]

It is sometimes useful to identify a set A of equations over a known signature σ' with the equational theory $\langle\sigma, \mathrm{A}\rangle$ where σ is the least subsignature of σ' for which A may be considered as a set of σ-equations. Then $\Sigma\mathrm{A} = \sigma$.

A closed term or equation is one in which no variables occur.

The action of signature morphisms is extended to terms, equations and theories in the natural way:
for a variable $\mathbf{x} = \langle \mathrm{x}, \mathrm{S}\rangle$: $\phi \cdot \mathbf{x} = \langle \mathrm{x}, \phi\cdot \mathrm{S}\rangle$,
$\phi\cdot \mathbf{f}\mathbf{t}_1 \ldots \mathbf{t}_{n-1} = (\phi\cdot\mathbf{f})(\phi\cdot\mathbf{t}_1)\ldots(\phi\cdot\mathbf{t}_{n-1})$,
$\phi\cdot(\mathbf{s}\mathbf{=}\mathbf{t}) = (\phi\cdot\mathbf{s})\mathbf{=}(\phi\cdot\mathbf{t})$, and
$\phi\cdot\langle\sigma, \mathrm{A}\rangle = \langle\phi\cdot\sigma, \phi\cdot\mathrm{A}\rangle$.

The equational theories form a semilattice with the supremum operation + defined by $\langle\sigma, \mathrm{A}\rangle + \langle\tau, \mathrm{B}\rangle = \langle\sigma\cup\tau, \mathrm{A}\cup\mathrm{B}\rangle$.

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[1] So *prima facie* a theory is a triple $\mathcal{A} = \langle\sigma, \mathrm{X}, \mathrm{A}\rangle$; but we abstract from the second component — ultimately because in equational logic we need variables only for the purpose of writing down nonlogical axioms.

4. Module algebra

The module algebra $BMA_0[eql]$ presented here differs in some respects from the module algebra $BMA[fol]$ of Bergstra c.s. On the one hand, a number of things, having to do with signatures in particular, will not be specified completely; on the other hand the parameter *eql* is less general than *fol*.

We will be concerned with three sorts: the sort *SIG* of signatures (with variables *x*, *y* en *z*), the sort *REN* of renamings (variable *r*) and the sort *M* of modules (with variables *X*, *Y* en *Z*). We have the following operations:

constants of sort *SIG*, in particular $\varnothing$ (the void signature)
$SIG \cup SIG \to SIG$ (union)
$REN \cdot SIG \to SIG$ (application of renaming)
$SIG\ /\ REN \to SIG$ (unaffected part of signature)
$SIG \cap SIG \to SIG$ (intersection)
constants of sort *M* (a constant $\langle\mathcal{A}\rangle$ for each equational theory $\mathcal{A}$)
$\Sigma : M \to SIG$ (signature)
$\mathrm{T} : SIG \to M$ (injection)
$REN \cdot M \to M$ (application of renaming)
$M + M \to M$ (combination)
$SIG \,\square\, M \to M$ (export/reduction)

The axioms of $BMA_0[eql]$ are listed below. We stipulate that unary operators bind more strongly than binary, and that $\square$ binds more strongly than +. Item (s1) represents a finite sequence of equations. The starred items are axiom schemes. I write $‹\sigma, \mathcal{A}›$ instead of $‹\langle\sigma, \mathcal{A}\rangle›$.

(s1)	$\langle SIG, \cup, \cap, \varnothing\rangle$ is a distributive lattice with least element $\varnothing$
(s2)	$(x / r) / r = x / r$
(s3)	$(x / r) \cup x = x$
(s4)	$(x \cup y)/ r = (x / r) \cup (y / r)$
(s5)	$(x \cap y)/ r = (x / r) \cap (y / r)$
(S1*)	$\Sigma‹\mathcal{A}› = \Sigma\mathcal{A}$
(S2)	$\Sigma \mathrm{T}x = x$
(S3)	$\Sigma(X+Y) = \Sigma X \cup \Sigma Y$
(S4)	$\Sigma(x \square Y) = x \cap \Sigma Y$
(S5)	$\Sigma(r\cdot Y) = r\cdot\Sigma Y$
(R1)	$r\cdot‹\sigma, \mathrm{A}› = ‹r\cdot\sigma, r\cdot\mathrm{A}›$
(R2)	$r\cdot\mathrm{T}x = \mathrm{T}(r\cdot x)$
(R3)	$r\cdot(X+Y) = r\cdot X + r\cdot Y$
(R4)	$r\cdot(x \square Y) = r\cdot x \square r\cdot Y$
(R5)	$r\cdot(r\cdot X) = X$
(R6)	$r\cdot((\sigma / r) \square X) = (\sigma / r) \square X$
(C1)	$X+Y = Y+X$
(C2)	$(X + Y) + Z = X + (Y + Z)$
(C3)	$\mathrm{T}(x \cup y) = \mathrm{T}x + \mathrm{T}y$
(C4)	$X + \mathrm{T}\Sigma X = X$
(C6*)	$‹\mathcal{A}› + ‹\mathcal{B}› = ‹\mathcal{A} + \mathcal{B}›$
(E1)	$\Sigma X \square X = X$
(E2)	$x \square (y \square Z) = (x \cap y) \square Z$
(E4)	$\Sigma X \square (X+Y) = X + \Sigma X \square Y$

Instead of using equational theories, with the schemes (S1) and (C6), we might have taken equations for constants, with a scheme $\Sigma(‹\alpha›) = \Sigma\{\alpha\}$.

For terms s and t of sort SIG I abbreviate $s = s \cap t$ to $s \subseteq t$. A model of $BMA_0[eql]$ will be called a *module algebra*. The word ‘model’ in this context must not be taken too strictly: the sorts may be proper classes, and analogously for the operations.

The axioms (C5) and (E3) of [BHK] are missing; they are not true in the interpretation to be given here (I will return to this point in §9). Our axioms (S1*), (R6) and (E4) differ from the axioms of the same names in [BHK]; furthermore we have an extra schema (C6*). The deviation in (R6) is a matter of notation. The deviation in (E4) on the other hand is a matter of substance. In the sequel the version of Bergstra c.s. will be called (E4i) (i for *implicative*); I shall prove that against the background of the rest of $BMA_0[eql]$, (E4) and (E4i) are equivalent. One direction is simple: (E4) is a special case of (E4i) [BHK, 3.1(7)].

The *common export operator* [BHK, §3.3] is defined by

(CE) $\quad X \,\square\, Y = (\Sigma(X) \cap \Sigma(Y)) \,\square\, (X+Y)$,

and the matching export distribution axiom is

(E4*) $\quad X \,\square\, Y = \Sigma(Y) \square X + \Sigma(X) \square Y$.

4.1 Theorem (Van Glabbeek). (E4*) holds in every module algebra.

Proof. $X \,\square\, Y = (\Sigma(X) \cap \Sigma(Y)) \,\square\, (X+Y)$ by definition (CE)

$= (\Sigma(X) \cap \Sigma(Y)) \,\square\, (\Sigma(X) \square (X+Y))$ (s1) and (E2)

$= (\Sigma(X) \cap \Sigma(Y)) \,\square\, (X + \Sigma(X) \square Y)$ (E4)

$= (\Sigma(X) \cap \Sigma(Y)) \,\square\, (X + (\Sigma(X) \cap \Sigma(Y)) \,\square\, Y)$ (E1) and (E2)

$= (\Sigma(X) \cap \Sigma(Y)) \,\square\, X + (\Sigma(X) \cap \Sigma(Y)) \,\square\, Y$ (E4) and (C1)

$= \Sigma Y \square X + \Sigma X \square Y$, by (s1), (E2) and (E1). ¶

In [BHK, §3.3] one finds a derivation of (E4i) from the other axioms and (E4*). This derivation uses a form of the axiom (E3):

(E3⁻) $\quad x \,\square\, (\mathrm{T}x + Y) = \mathrm{T}x + x \square Y$.

We do not have (E3) at our disposal, but it is easily seen that (E3⁻) is a special case of our (E4). We conclude that (E4i) holds in all module algebras:

4.2 Theorem. In every module algebra

$$x \subseteq \Sigma X \cap \Sigma Y \;\Rightarrow\; x \,\square\, (X+Y) = x \square X + x \square Y .$$

It is easily checked that the present system of module algebra is strong enough for the proof of the normal form theorem of [BHK, §3.2] to go through. That is to say, adding some applications of the scheme (C6*) we get

4.3 Proposition. Every closed module expression is equivalent to an expression of the form $\sigma \square \langle \mathcal{A} \rangle$, for a signature σ and an equational theory $\mathcal{A}$.

Indeed, inspection of the proof immediately gives a more effective version:

4.4 Corollary. Every closed module expression that has been constructed without the use of constants representing infinite theories is equivalent to an expression of the form $\sigma \square \langle \mathcal{A} \rangle$, for a signature σ and a finite equational theory $\mathcal{A}$.

With respect to the signatures and renamings proposed in §2, (s1)-(s5) are evidently true. In the sequel we shall develop a notion of modules, such that we obtain a model for the entire theory.

5. Algebras

Let σ be a signature. A σ-*algebra* is an assignment $\mathbf{A}$ of a nonvoid set (sort) $\mathbf{A}(\mathrm{S})$ to each $\mathrm{S} \in \mathrm{S}\sigma$ and a function

$$\mathbf{A}(\mathbf{f}) : \mathbf{A}(\mathrm{S}_1) \times \ldots \times \mathbf{A}(\mathrm{S}_n) \to \mathbf{A}(\mathrm{S}_0)$$

to each $\mathbf{f} = \langle \mathrm{f}, \mathrm{S}_1, \ldots, \mathrm{S}_n, \mathrm{S}_0 \rangle$ in $\mathrm{F}\sigma$. If there is no need to mention $\mathbf{A}$, I write f instead of $\mathbf{A}(\mathbf{f})$.

By the stipulation that sorts are nonvoid, our many-sorted algebra becomes a direct generalization of the standard single-sorted universal algebra that one finds in Grätzer's book [Gr]. Meseguer and Goguen describe an alternative approach in [GM] and [MG].

An *algebra* is a σ-algebra for some signature σ. Let $\mathbf{A}$ be an algebra; then $\Sigma\mathbf{A}$, the signature of $\mathbf{A}$, is the (unique) signature σ such that $\mathbf{A}$ is a σ-algebra. Algebras $\mathbf{A}$ and $\mathbf{B}$ are *similar* if $\Sigma\mathbf{A} = \Sigma\mathbf{B}$. For $\upsilon \subseteq \Sigma\mathbf{A}$, the υ-*reduct* $\mathbf{A}|\upsilon$ of $\mathbf{A}$ is the υ-algebra $\mathbf{B}$ defined by $\mathbf{B}(\mathbf{d}) = \mathbf{A}(\mathbf{d})$ for all $\mathbf{d} \in \upsilon$; $\mathbf{A}$ is called an *expansion* of $\mathbf{B}$. For any signature τ the τ-*reduct* of $\mathbf{A}$ is defined by

$$\tau \square \mathbf{A} = \mathbf{A} | (\Sigma\mathbf{A} \cap \tau).$$

An arbitrary algebra $\mathbf{C}$ is τ-*generated* if every $\mathbf{f} \in \mathrm{F}\tau$ whose domain sorts belong to $\Sigma\mathbf{C}$, itself belongs to $\Sigma\mathbf{C}$.

Let I be a set. An I-*sorted set* is a family $X = \langle X_i | i \in I \rangle$ of sets. If σ is a signature, then 'σ-*sorted set* is short for 'Sσ-sorted set'. Many notions and constructions relating to sets have immediate componentwise generalisations to sorted sets. For instance, if A and B are both I-sorted, then

$$A \cup B = \langle A_i \cup B_i \,|\, i \in I \rangle,$$

and $A \subseteq B$ if and only if $A_i \subseteq B_i$ for all $i \in I$. An I-sorted function f from A to B is a family $\langle f_i \,|\, i \in I \rangle$ of functions $f_i : A_i \to B_i$. The I-sorted set X that has $X_i = \varnothing$ for all $i \in I$ will be denoted by $\varnothing$, when I is sufficiently understood. An *equivalence relation* on an I-sorted set X is a family $\Theta = \langle \Theta_i | i \in I \rangle$ of equivalence relations Θ_i on X_i. Usually the index will be suppressed: given $x, y \in X_i$ we write

$\langle x,y\rangle \in \Theta$, $x\Theta y$ or $x \equiv y \pmod{\Theta}$ instead of $\langle x,y\rangle \in \Theta_i$ and such.[1] The *equivalence class* of $x \in X_i$ is $\Theta_i\{x\}$, usually written $\Theta\{x\}$.

Let **A** and **B** be algebras. Then **A** is a *subalgebra* of **B**, notation $\mathbf{A} \leq \mathbf{B}$, if $\Sigma\mathbf{A} = \Sigma\mathbf{B}$, $\mathbf{A}(\mathrm{S}) \subseteq \mathbf{B}(\mathrm{S})$ for each $\mathrm{S} \in \mathrm{S}\Sigma\mathbf{A}$, and for all $\mathbf{f} = \langle \mathrm{f}, \mathrm{S}_1, \ldots, \mathrm{S}_n \rangle \in \Sigma\mathbf{A}$, $\mathbf{A}(\mathbf{f}) = \mathbf{B}(\mathbf{f})|(\mathbf{A}(\mathrm{S}_1) \times \ldots \times \mathbf{A}(\mathrm{S}_{n-1}))$. I call **A** a *subreduct* of **B**, notation $\mathbf{A} \subseteq \mathbf{B}$, if **A** is a subalgebra of a reduct of **B**; the algebra **B** is then a *superexpansion* of **A**. Finally, **A** is a *generated subreduct* of **B** if $\mathbf{A} \subseteq \mathbf{B}$ and **A** is $\Sigma\mathbf{B}$-generated. The $\Sigma\mathbf{A}$-sorted set $\mathbf{A}|\mathrm{S}\Sigma\mathbf{A}$ (the *universe* or *carrier* of **A**) will be denoted by the corresponding italic letter A. A *subuniverse* of **A** is a $\Sigma\mathbf{A}$-sorted subset of A that is closed under the operations of **A**. An intersection of subuniverses of **A** is again a subuniverse of **A**. Let X be a subset of A; the least subuniverse of **A** that contains X, the subuniverse *generated by* X will be denoted by $\langle X \rangle_{\mathbf{A}}$. The subuniverse $\langle X \rangle_{\mathbf{A}}$ is the carrier of a generated subreduct $\mathbf{A}[X]$ of **A**, the operations of which are the restrictions to $\langle X \rangle_{\mathbf{A}}$ of the operations of **A**. (Observe that $\Sigma(\mathbf{A}[X])$ may be properly contained in $\Sigma\mathbf{A}$.)

A sort S in a signature σ is *inhabited* if there exist closed terms over σ of sort S. A signature is (*fully*) *inhabited* if all its sorts are inhabited.

The *algebra of closed terms* over a signature σ is the algebra $\mathbf{T}_\sigma$ that has for sorts the inhabited sorts of σ, and for elements the closed terms over σ, with functions

$$\langle \mathbf{t}_1, \ldots, \mathbf{t}_n \rangle \mapsto \mathbf{f}\mathbf{t}_1 \ldots \mathbf{t}_n.$$

Observe that $\Sigma(\mathbf{T}_\sigma) = \sigma$ only if σ is fully inhabited; a function declaration **f** belongs to $\Sigma(\mathbf{T}_\sigma)$ if and only if all its domain sorts are inhabited.

A σ-homomorphism from a σ-algebra **A** to a σ-algebra **B** is een σ-sorted function $\phi : A \to B$ that commutes with the σ-operations, i.e. such that

$$\phi(\mathbf{A}(\mathbf{g})(a_1, \ldots, a_n)) = \mathbf{B}(\mathbf{g})(\phi(a_1), \ldots, \phi(a_n))$$

for all $a_1, \ldots, a_n \in A$ that come into consideration. The category of σ-algebras with σ-homomorphisms will be denoted by $\mathbf{Alg}_\sigma$. For arbitrary algebras **A** and **B** a *homomorphism* from **A** to **B** is a $\Sigma\mathbf{A}$-homomorphism from **A** to $\mathbf{B}|\Sigma\mathbf{A}$ (it follows that such a homomorphism can exist only if $\Sigma\mathbf{B} \supseteq \Sigma\mathbf{A}$). The category of all algebras and homomorphisms will be denoted by **Alg**. If **A** is an algebra and $\phi : \mathbf{B} \to \mathbf{A}$ a homomorphism, then the subreduct of **A** *generated by* ϕ is the subreduct $\phi(\mathbf{B}) := \mathbf{A}[\phi(B)]$. More general: if for every $i \in I$, $\phi_i : \mathbf{B}_i \to \mathbf{A}$ is a homomorphism, then $\mathbf{A}[\bigcup_i \phi_i(B)]$ is the subreduct of **A** generated by $\{\phi_i\}_{i \in I}$.

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[1]In principle, this notation is ambiguous, because we have not ruled out that $x, y \in X_i \cap X_j$, with $i \neq j$, $\langle x,y\rangle \in \Theta_i$ and $\langle x,y\rangle \notin \Theta_i$. When we write that x and y are equivalent modulo Θ, we must assume we have a definite sort in mind that x and y belong to.

Let $\mathbf{A}$ be a σ-algebra. The interpretation $\mathbf{A}_X(\mathbf{t})$ of a σ-term $\mathbf{t}$ in variables from X is a function from $\prod(\mathbf{A}(S_\mathbf{x})|\mathbf{x}\in X)$ to $\mathbf{A}(S_\mathbf{t})$, inductively defined by: (i) if $\mathbf{t}\in X$, then $\mathbf{A}_X(\mathbf{t})$ is the projection $\pi_\mathbf{t}:\prod(\mathbf{A}(S_\mathbf{x})|\mathbf{x}\in X)\to\mathbf{A}(S_\mathbf{t})$; (ii) if $\mathbf{t} = \mathbf{f}\mathbf{t}_1\ldots\mathbf{t}_n$, then $\mathbf{A}_X(\mathbf{t}) = \mathbf{A}(\mathbf{f})\circ(\mathbf{A}_X(\mathbf{t}_1),\ldots,\mathbf{A}_X(\mathbf{t}_n))$. The subscript X will be suppressed whenever possible. If there is no need to mention $\mathbf{A}$, I write *t* instead of $\mathbf{A}(\mathbf{t})$. Introduction of $\mathbf{t}$ as $\mathbf{t}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ is meant to imply that $\mathbf{x}_1,\ldots,\mathbf{x}_n$ are all distinct and $\mathbf{t}$ is a term in variables from $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$; so given a suitable sequence $a_1,\ldots,a_n$ of elements of $\mathbf{A}$ we may write $t(a_1,\ldots,a_n)$ for the value $\mathbf{A}_X(\mathbf{t})(\alpha)$ of $\mathbf{t}$ in $\mathbf{A}$ under any assignment α to the variables that maps $\mathbf{x}_i$ to a_i, $1\leq i\leq n$. It is easy to see that if $\mathbf{s}$ and $\mathbf{t}$ are terms in $X\cap Y$, $\mathbf{A}_X(\mathbf{s}) = \mathbf{A}_X(\mathbf{t})$ if and only if $\mathbf{A}_Y(\mathbf{s}) = \mathbf{A}_Y(\mathbf{t})$. In other words, in these contexts the notation $\mathbf{A}(\mathbf{s})$ is sufficient. The equation $\mathbf{s{=}t}$ is *valid* in $\mathbf{A}$, notation $\mathbf{A}\vDash\mathbf{s{=}t}$, if $\mathbf{A}(\mathbf{s}) = \mathbf{A}(\mathbf{t})$. If $\mathbf{A}$ is a σ-algebra and $\mathbf{s}$ and $\mathbf{t}$ are terms over a signature $\tau\supset\sigma$, then $\mathbf{A}\vDash\mathbf{s{=}t}$ if $\mathbf{B}(\mathbf{s}) = \mathbf{B}(\mathbf{t})$ for every expansion $\mathbf{B}$ of $\mathbf{A}$ to a τ-algebra. The notations $\mathrm{A}\vDash\mathrm{B}$ (for a set B of equations), $\boldsymbol{K}\vDash\beta$ (for a class $\boldsymbol{K}$ of algebras) and $\boldsymbol{K}\vDash\mathrm{B}$ are defined in the usual way.

An algebra $\mathbf{A}$ is a *model* of a theory $\mathcal{B}=\langle\sigma,\mathrm{B}\rangle$ if $\sigma = \Sigma\mathbf{A}$ and $\mathbf{A}\vDash\mathrm{B}$. The class of all models of $\mathcal{B}$ (the *model class* of $\mathcal{B}$ or the *variety determined by* $\mathcal{B}$) I denote by $Mod(\mathcal{B})$. In particular the elements of a variety are similar. The common signature of the elements of a nonvoid class $\boldsymbol{K}$ of similar algebras will be denoted as $\Sigma(\boldsymbol{K})$. A σ-variety is a variety of algebras of signature σ. If $\boldsymbol{K}$ is a class of similar algebras, and X a set of variables for $\Sigma(\boldsymbol{K})$, then the *(equational) theory* of $\boldsymbol{K}$ in X is the equational theory $\mathcal{E}q_X(\boldsymbol{K})$ determined by the set of equations over $\Sigma(\boldsymbol{K})$ in variables from X that are valid in $\boldsymbol{K}$. If X and Y are both sortwise infinite, then $\mathcal{E}q_X(\boldsymbol{K})$ and $\mathcal{E}q_Y(\boldsymbol{K})$ are equivalent (all axioms of $\mathcal{E}q_Y(\boldsymbol{K})$ are substitution instances of axioms of $\mathcal{E}q_X(\boldsymbol{K})$ and vice versa). For this reason the variables will usually not be mentioned: $\mathcal{E}q(\boldsymbol{K})$ represents an arbitrary $\mathcal{E}q_X(\boldsymbol{K})$ with sortwise infinite X. The turnstile symbol $\vDash$ will also be used for the (semantic) relation of logical consequence: $\mathrm{A}\vDash\beta$ if β is valid in every algebra in which all the elements of A are valid. (Similarly $\mathrm{A}\vDash\mathrm{B}$, etc.) It is well known that a finitary deduction system exists for equational logic; so if $\mathrm{A}\vDash\beta$, there is a finite $\mathrm{A}_0\subseteq\mathrm{A}$ such that $\mathrm{A}_0\vDash\beta$.

Let $\boldsymbol{K}$ be a class of similar algebras. An algebra $\mathbf{A}$ is $\boldsymbol{K}$*-generated*, or *generated over* $\boldsymbol{K}$, if $\Sigma\boldsymbol{K}\Box\mathbf{A}$ is a generated subreduct of an element of $\boldsymbol{K}$. For an equational theory $\mathcal{B}$, '$Mod(\mathcal{B})$-generated' will be abbreviated to '$\mathcal{B}$-generated'; and clearly, $\boldsymbol{K}(\sigma)$-generated is the same as σ-generated. The class of all $\boldsymbol{K}$-generated algebras will be denoted by $\boldsymbol{G}(\boldsymbol{K})$.

5.1 Lemma. Let $\boldsymbol{K}$ be a variety. Then $\boldsymbol{G}(\boldsymbol{K})$, considered as a full subcategory of **Alg**, has an initial object.

Proof. Let $\sigma = \Sigma\boldsymbol{K}$; and let τ be the largest inhabited subsignature of σ. Then $\Sigma(\mathbf{T}_\sigma) = \tau$. The σ-sorted relation Θ over T_σ defined by

$$\langle \mathbf{s}, \mathbf{t} \rangle \in \Theta \Leftrightarrow \boldsymbol{K} \vDash \mathbf{s} = \mathbf{t}$$

is a congruence of $\mathbf{T}_\sigma$. Let $\mathbf{A} = \mathbf{T}_\sigma/\Theta$. Obviously there exists exactly one homomorphism from $\mathbf{A}$ to any element of $\boldsymbol{G}(\boldsymbol{K})$. Moreover, by a similar construction on an expansion σ' of σ by constants of uninhabited sorts, we get, after reduction to σ, an element $\mathbf{B}$ of $\boldsymbol{K}$, such that $\mathbf{A} \leq \mathbf{B}|\tau$; so $\mathbf{A} \in \boldsymbol{G}(\boldsymbol{K})$, and $\mathbf{A}$ is initial in $\boldsymbol{G}(\boldsymbol{K})$. ¶

The initial $\boldsymbol{K}$-generated algebra will be denoted by $\mathbf{I}_{\boldsymbol{K}}$; for an equational theory $\mathcal{C}$ I write $\mathbf{I}_{\mathcal{C}}$ instead of $\mathbf{I}_{Mod(\mathcal{C})}$. We have $\mathbf{I}_{\boldsymbol{K}} \in \boldsymbol{K}$ if and only if $\Sigma\boldsymbol{K}$ is inhabited.

The expansion of an algebra $\mathbf{A}$ by declarations of proper names for the elements of a subset X of its universe will be denoted by $\mathbf{A}_X$. These declarations are of the form $\mathbf{x} = \langle x, \mathrm{S} \rangle$, with $x \in X_\mathrm{S}$. They are not supposed to belong to D(N), and I assume that the proper name x is invariant under D-bijections, so $\phi \cdot \mathbf{x} = \langle x, \phi \cdot \mathrm{S} \rangle$. Furthermore, proper names of elements of distinct algebras never just *happen* to coincide; they only do so if they name the same thing *and* the one algebra has been specified to be a subreduct of the other. I use the name X as a name of the set of all declarations $\langle x, \mathrm{S} \rangle$.

The *diagram* $\Delta\mathbf{A}$ of an algebra $\mathbf{A}$ is the set of all closed equations that are true in $\mathbf{A}_A$.

Let $\boldsymbol{K}$ be a σ-variety and $\mathbf{A}$ an arbitrary algebra. Let $\mathcal{T} = \mathcal{E}q\boldsymbol{K} \cup \Delta\mathbf{A}$. The *$\boldsymbol{K}$-generated superexpansion of* $\mathbf{A}$ is defined by

$$F_{\boldsymbol{K}}(\mathbf{A}) = (\sigma \cup \Sigma\mathbf{A}) \,\Box\, \mathbf{I}_{\mathcal{T}}.$$

The mapping $a \mapsto \mathbf{I}_{\mathcal{T}}(\mathbf{a})$ determines a homomorphism $\eta_{\boldsymbol{K},\mathbf{A}} : \mathbf{A} \to F_{\boldsymbol{K}}(\mathbf{A})$. This homomorphism has the following universal property: if $\mathbf{B}$ is $\boldsymbol{K}$-generated, then every homomorphism $\phi : \mathbf{A} \to \mathbf{B}$ factors uniquely through $\eta_{\boldsymbol{K},\mathbf{A}}$. For suppose $\phi : \mathbf{A} \to \mathbf{B}$; then $(\mathbf{B}, \phi)$, the expansion of $\mathbf{B}$ by the declarations in A interpreted through ϕ, is $\mathcal{T}$-generated. Now let $\psi : F_{\boldsymbol{K}}(\mathbf{A}) \to \mathbf{B}$ be determined by the unique homomorphism $\mathbf{I}_{\mathcal{T}} \to (\mathbf{B}, \phi)$; then $\phi = \psi \circ \eta$, and this factorization is unique because η generates $F_{\boldsymbol{K}}(\mathbf{A})$.

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\eta"] \arrow[dr, "\phi"'] & F_{\boldsymbol{K}}\mathbf{A} \arrow[d, "\psi"] \\ & \mathbf{B} \end{tikzcd} \tag{5.1}$$

In the special case that $\mathbf{A} = F_{\boldsymbol{K}}(\mathbf{C})$ and $\phi = 1_{\mathbf{A}}$ I will denote ψ by $\kappa_{\boldsymbol{K},\mathbf{C}}$.

The operation $F_{\boldsymbol{K}}$ becomes a functor, and $\eta_{\boldsymbol{K}}$ a natural transformation

$$1_{\mathbf{Alg}} \to F_{\boldsymbol{K}} \,,$$

when we define $F_{\boldsymbol{K}}(\phi)$, for $\phi : \mathbf{A} \to \mathbf{B}$, as the unique $\psi : F_{\boldsymbol{K}}(\mathbf{A}) \to F_{\boldsymbol{K}}(\mathbf{B})$ such that $\eta_{\boldsymbol{K},\mathbf{B}} \circ \phi = \psi \circ \eta_{\boldsymbol{K},\mathbf{A}}$. If $\boldsymbol{K} = Mod(\mathcal{C})$, we write $F_{\mathcal{C}}$ and $\eta_{\mathcal{C}}$ instead of $F_{Mod(\mathcal{C})}$ and $\eta_{Mod(\mathcal{C})}$. Observe that $\mathbf{I}_{\boldsymbol{K}} = F_{\boldsymbol{K}}(\varnothing)$. I abbreviate $F_{\langle\sigma,\varnothing\rangle}$ to T_σ,[1] and $\eta_{\langle\sigma,\varnothing\rangle}$ to η_σ.

Obvious variations on standard arguments would now lead to an analogue of the well-known theory of free single-sorted algebras, proving, for example, that if $\boldsymbol{K}$ and $\boldsymbol{L}$ are σ-varieties, $\boldsymbol{K} = \boldsymbol{L}$ if and only if $F_{\boldsymbol{K}}(X) = F_{\boldsymbol{L}}(X)$ for some sort-wise infinite σ-sorted set X.

5.2 Proposition. Let $\boldsymbol{K}$ be a σ-variety, $F = F_{\boldsymbol{K}}$ and $\eta = \eta_{\boldsymbol{K}}$. Then
(i) ηF has a left inverse;
(ii) $\eta F = F\eta$.

Proof. (i) For every algebra $\mathbf{C}$, let $\kappa_{\boldsymbol{K},\mathbf{C}}$ be defined as above. The naturality of $\kappa_{\boldsymbol{K}} =: \kappa$ is seen as follows. Suppose $\phi : \mathbf{A} \to \mathbf{B}$. By naturality of η and by definition of κ,

$$\kappa_{\mathbf{B}} \circ FF\phi \circ \eta_{F\mathbf{A}} = \kappa_{\mathbf{B}} \circ \eta_{F\mathbf{B}} \circ F\phi = F\phi = F\phi \circ \kappa_{\mathbf{A}} \circ \eta_{F\mathbf{A}} \,.$$

By the universal property of $\eta_{F\mathbf{A}}$ this implies $\kappa_{\mathbf{B}} \circ FF\phi = F\phi \circ \kappa_{\mathbf{A}}$.
(ii) By definition $F\eta_{\mathbf{A}}$ is the unique $\psi : F\mathbf{A} \to FF\mathbf{A}$ such that $\eta_{F\mathbf{A}} \circ \eta_{\mathbf{A}} = \psi \circ \eta_{\mathbf{A}}$; so $F\eta_{\mathbf{A}} = \eta_{F\mathbf{A}}$. ¶

5.3 Definition. Let ϕ be a D-bijection and $\mathbf{A}$ an algebra. Then $\phi^*(\mathbf{A}) = \mathbf{A} \circ \phi$.

5.4 Proposition. $\Sigma(\phi^*(\mathbf{A})) = \phi^{-1} \cdot \Sigma\mathbf{A}$.

Let ψ be a homomorphism from $\mathbf{A}$ to $\mathbf{B}$. Define a $(\phi^{-1} \cdot \Sigma\mathbf{A})$-sorted mapping $\phi^*(\psi)$ by: $(\phi^*(\psi))_S = \psi_{\phi(S)}$. Then $\phi^*(\psi)$ is a homomorphism from $\phi^*(\mathbf{A})$ to

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[1] So $\mathbf{T}_\sigma = T_\sigma(\varnothing)$. In principle this invites confusion with the universe T_σ of the term algebra, but we may hope that the context will make clear what is meant.

$\phi^*(\mathbf{B})$; and ϕ^* is an automorphism of **Alg**. On a higher level, the assignment $\phi \mapsto \phi^*$ defines an embedding of the group of D-bijections into the group of automorphisms of **Alg**.[1]

We will take *modules*, essentially, to be uniform expansions $\eta_{\boldsymbol{K}}$ over varieties $\boldsymbol{K}$. However, we will go to considerable lengths now in order to embed these expansions into a notion that seems to be more general; in a way analogous to the consideration of model classes instead of elementary classes in [BHK].

[1] The carrier of the latter group is of course a proper class.

6. Uniformities

A functor will be called *continuous* if it preserves direct limits, that is, colimits of functors from a directed partial ordering. (Henceforth I shall speak of *ordering*, ‘partial’ being a rudiment from that golden age when total order was the norm.)

6.1 Definition. A *uniformity* in a category **C** is a natural transformation η from the identity functor $1_{\mathbf{C}}$ to a continuous endofunctor F of **C**, with the properties
(i) $\eta F = F\eta$, and
(ii) $F\eta$ has a left inverse.

6.2 Lemma. Let **C** be a category, F an endofunctor of **C**, $\eta : 1_{\mathbf{C}} \xrightarrow{\bullet} F$, and $\eta F = F\eta$.
(a) If ηF has a left inverse, this is also a right inverse, so $FF \cong F$.
(b) If κ is an inverse of ηF, then $\kappa F = F\kappa$.

Proof. (a) Suppose $\kappa \bullet \eta F = F$. Then

$$\begin{aligned} \eta F \bullet \kappa &= F\kappa \bullet \eta FF && \text{by naturality of } \eta \text{ (cf. diagram 6.1)} \\ &= F\kappa \bullet F\eta F = F \circ (\kappa \bullet \eta F) = FF. \end{aligned}$$

$$\begin{tikzcd} FFc \arrow[r, "\eta_{FFc}"] \arrow[d, "\kappa_c"'] & FFFc \arrow[d, "F\kappa_c"] \\ Fc \arrow[r, "\eta_{Fc}"'] & FFc \end{tikzcd}$$

(6.1)

(b) By repeated use of the exchange law:

$$F\kappa = F\kappa \bullet FFF = F\kappa \bullet ((\eta F \bullet \kappa)F) = F\kappa \bullet \eta FF \bullet \kappa F = F\kappa \bullet F\eta F \bullet \kappa F$$
$$= (F \circ (\kappa \bullet \eta F)) \bullet \kappa F = FF \bullet \kappa F = \kappa F \ . \qquad ¶$$

6.3 Examples.

6.3.1. In any category $\mathbf{C}$, $1_{\mathbf{C}} : 1_{\mathbf{C}} \to 1_{\mathbf{C}}$ is a uniformity.

6.3.2. Let $\boldsymbol{K}$ be a variety, $F = F_{\boldsymbol{K}}$ and $\eta = \eta_{\boldsymbol{K}}$. Then F is an endofunctor of **Alg**, and $\eta : 1 \to F$, by the definition of $F_{\boldsymbol{K}}$ in §5.

Let $\mathbf{I} = \langle I, \leq \rangle$ be a directed ordering, and $L : \mathbf{I} \to \mathbf{Alg}$. Suppose $\phi : L \stackrel{\bullet}{\to} \mathbf{A}$ is a colimit cone. Then $F\phi : FL \stackrel{\bullet}{\to} F\mathbf{A}$ is a cocone; let $\psi : FL \stackrel{\bullet}{\to} \mathbf{B}$ be another cocone. Define $g : F\mathbf{A} \to \mathbf{B}$ by

(1) $\quad g \bullet \eta_{\mathbf{A}} \bullet \phi = \psi \bullet \eta L$.

We must show that (1) is a correct definition. The algebra $F\mathbf{A}$ is generated by $\eta_{\mathbf{A}}$, and $\mathbf{A}$ by $\{\phi_i | i \in I\}$; as a consequence, every element of $F\mathbf{A}$ may be written in the form

$$(F\mathbf{A})(\mathbf{t})(\eta_{\mathbf{A}}\phi_{i_1}(x_1), \ldots, \eta_{\mathbf{A}}\phi_{i_n}(x_n)),$$

for a term $\mathbf{t}$ over $\Sigma F\mathbf{A}$, certain $i_1, \ldots, i_n \in I$, and elements x_1 of $Li_1, \ldots, x_n$ of Li_n. We must show that if

(2) $a = (F\mathbf{A})(\mathbf{s})(\eta_{\mathbf{A}}\phi_{i_1}(x_1), \ldots, \eta_{\mathbf{A}}\phi_{i_n}(x_n)) = (F\mathbf{A})(\mathbf{t})(\eta_{\mathbf{A}}\phi_{i_1}(x_1), \ldots, \eta_{\mathbf{A}}\phi_{i_n}(x_n))$,

$g(a)$ is the same by (1) on either representation of a. Let $\phi_{i_j}(x_j) =: a_j$, for $1 \leq j \leq n$. We conclude from (2) that

(3) $\quad \mathcal{E}q\boldsymbol{K} \cup \Delta\mathbf{A} \vDash \mathbf{s}(\mathbf{a}_1, \ldots, \mathbf{a}_n) \mathbf{=} \mathbf{t}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$.

Then for some finite $\Delta_0 \subseteq \Delta\mathbf{A}$,

(4) $\quad \mathcal{E}q\boldsymbol{K} \cup \Delta_0 \vDash \mathbf{s}(\mathbf{a}_1, \ldots, \mathbf{a}_n) \mathbf{=} \mathbf{t}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$.

We may assume that the elements of Δ_0 are of the form $\mathbf{b} \mathbf{=} \mathbf{f}\mathbf{c}_1 \ldots \mathbf{c}_k$, for parameters $b, c_1, \ldots, c_k \in A$. For each of these equations, some $i \in I$ exists with $y, z_1, \ldots, z_k$ in Li such that $b = \phi_i(y)$, $c_l = \phi_i(z_l)$ $(1 \leq l \leq k)$, and

$$y = (Li)(\mathbf{f})(z_1, \ldots, z_k).$$

Since $\mathbf{I}$ is directed, we may choose the same i for all the equations in Δ_0, even with $i \geq i_1, \ldots, i_n$; say $w_j = F(i_j \leq i)(x_j)$ $(1 \leq j \leq n$; by '$i_j \leq i$' I refer to the unique arrow $i_j \to i$ in I). We may also assume without loss of generality that the proper names in Δ_0 are contained in $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$; say $\Delta_0 = \Delta_0(\mathbf{a}_1, \ldots, \mathbf{a}_n)$, with all occurrences of proper names accounted for. Then by (4) and the truth of $\Delta_0(\mathbf{w}_1, \ldots, \mathbf{w}_n)$ in $(Li)_{\{w_1, \ldots, w_n\}}$:

(5) $\quad (FLi)(\mathbf{s})(\eta_{Li}(w_1),\ldots,\eta_{Li}(w_n)) = (FLi)(\mathbf{t})(\eta_{Li}(w_1),\ldots,\eta_{Li}(w_n))$,

so $g((F\mathbf{A})(\mathbf{s})(\eta_{\mathbf{A}}\phi_{i_1}(x_1),\ldots,\eta_{\mathbf{A}}\phi_{i_n}(x_n))) = g((F\mathbf{A})(\mathbf{s})(\eta_{\mathbf{A}}\phi_i(w_1),\ldots,\eta_{\mathbf{A}}\phi_i(w_n)))$

$$= \mathbf{B}(\mathbf{s})(g\eta_{\mathbf{A}}\phi_i(w_1),\ldots,g\eta_{\mathbf{A}}\phi_i(w_n)))$$

$$= \mathbf{B}(\mathbf{s})(\psi_i\eta_{Li}(w_1),\ldots,\psi_i\eta_{Li}(w_n)))$$

$$= \psi_i((FLi)(\mathbf{s})(\eta_{Li}(w_1),\ldots,\eta_{Li}(w_n)))$$

$$= \psi_i((FLi)(\mathbf{t})(\eta_{Li}(w_1),\ldots,\eta_{Li}(w_n)))$$

$$= g((F\mathbf{A})(\mathbf{t})(\eta_{\mathbf{A}}\phi_{i_1}(x_1),\ldots,\eta_{\mathbf{A}}\phi_{i_n}(x_n))).$$

So (1) is correct.

We have

(6) $\quad g\bullet F\phi \bullet\eta L = g\eta_{\mathbf{A}}\bullet\phi = \psi\bullet\eta L$

by naturality of η, hence, each FLi being generated by η_{Li},

(7) $\quad g\bullet F\phi = \psi$.

Conversely (1) follows from (7), so (7) determines g uniquely. So F is continuous.

Now by proposition 5.2, η_K is a uniformity.

6.3.3. Let $\mathbf{C}$ be a category, and $\eta : 1_{\mathbf{C}} \to F$ a uniformity in $\mathbf{C}$. Let $F^2 = F\circ F$, $\eta^2 = \eta\circ\eta$ etc. Then η^2 is a uniformity. In particular, if κ is the inverse of $F\eta$,

$$\kappa^2\bullet F^2\eta^2 = \kappa^2\bullet(F\eta \circ F\eta) = (\kappa\bullet F\eta)\circ(\kappa\bullet F\eta) = F^2.$$

6.3.4. Let ϕ be a D-bijection. Since ϕ^* is an automorphism of **Alg**, ϕ^* is continuous. Let $\eta : 1 \to F$ be a uniformity in **Alg**. Define $\phi\cdot X$ as $(\phi^{-1})^*\circ X\circ\phi^*$, for a functor or natural transformation X. Then $\phi\cdot F$, as a composition of continuous functors, is continuous. Furthermore $(\phi^{-1})^*\circ 1_{\mathbf{Alg}}\circ\phi^* = 1_{\mathbf{Alg}}$, so

$$\phi\cdot\eta : 1_{\mathbf{Alg}} \to \phi\cdot F\,.$$

Now $\phi\cdot(X\circ Y) = (\phi^{-1})^*\circ X\circ Y\circ\phi^* = (\phi^{-1})^*\circ X\circ\phi^*\circ(\phi^{-1})^*\circ Y\circ\phi^* = (\phi\cdot X)\circ(\phi\cdot Y)$, so $(\phi\cdot\eta)\circ(\phi\cdot F) = \phi\cdot\eta F = \phi\cdot F\eta = (\phi\cdot F)\circ(\phi\cdot\eta)$. For the other composition we have by the exchange law:

$$\phi\cdot(X\bullet Y) = (\phi^{-1})^*\circ(X\bullet Y)\circ\phi^* = ((\phi^{-1})^*\circ X\circ\phi^*)\bullet((\phi^{-1})^*\circ Y\circ\phi^*) = (\phi\cdot X)\bullet(\phi\cdot Y);$$

so if κ is the left inverse of $F\eta$,

$$(\phi\cdot\kappa)\bullet((\phi\cdot F)\circ(\phi\cdot\eta)) = \phi\cdot(\kappa\bullet F\eta) = \phi\cdot 1 = 1.$$

We conclude that $\phi\cdot\eta$ is a uniformity in **Alg**.

The formulas of signature change derived above may be summed up as follows:

6.4 Proposition. Let ϕ be a D-bijection, and X, Y properly composable functors or natural transformations in **Alg**. Let © be either $\circ$ or $\bullet$; then

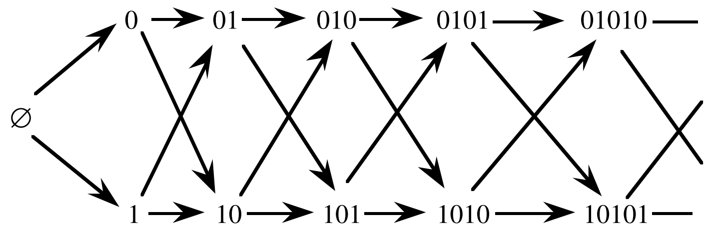
$$\phi\cdot(X © Y) = (\phi\cdot X) © (\phi\cdot Y).$$

6.5 Definition. Let $\eta : 1_{\mathbf{C}} \to F$ and $\theta : 1_{\mathbf{C}} \to G$ be uniformities in a category $\mathbf{C}$. A *morphism* from η to θ is a natural transformation $\alpha : F \xrightarrow{\bullet} G$ with the property that $\theta = \alpha\bullet\eta$.

The category of uniformities in a category $\mathbf{C}$ that we obtain in this way I call $\mathbf{Un}(\mathbf{C})$.

6.6 Examples. Let $\eta : 1_{\mathbf{C}} \to F$ be a uniformity in a category $\mathbf{C}$.
6.6.1. Observe that $\eta : 1_{\mathbf{C}} \to \eta$ in $\mathbf{Un}(\mathbf{C})$. It is evident that η is the *only* morphism from $1_{\mathbf{C}}$ to η. So $1_{\mathbf{C}}$ is initial in $\mathbf{Un}(\mathbf{C})$.
6.6.2. Let $\kappa : F^2 \to F$ be the inverse of ηF. Then $\kappa : \eta^2 \to \eta$, for $\kappa\bullet\eta^2 = \kappa\bullet F\eta\bullet\eta = F\bullet\eta = \eta$ since $\eta^2 = (F\bullet\eta)\circ(\eta\bullet 1_{\mathbf{C}}) = F\eta\bullet\eta$. Conversely $\eta F : F \to F^2$, and κ and ηF are inverses in $\mathbf{Un}(\mathbf{C})$.

6.7 Definition. Let $\eta : 1_{\mathbf{C}} \to F$ and $\theta : 1_{\mathbf{C}} \to G$ be uniformities in a category $\mathbf{C}$. Let $P \subseteq \{0,1\}^*$ be the set of those sequences of zeros and ones in which adjoining digits are distinct, and $\mathbf{P}$ the ordering $\langle P, \leq\rangle$ in which $p < q$ if and only if $|p| < |q|$ (see diagram 6.2; the symbol $\varnothing$ denotes the empty sequence).



(6.2)

Take an object c in $\mathbf{C}$. For $p \in P$ we define $(F\#_{\eta,\theta}G)(c)(p) = c_p$ as follows:
$c_{\varnothing} = c$; $c_{p0} = Fc_p$, $c_{p1} = Gc_p$.
If q covers p in $\mathbf{P}$ (notation $q \succ p$), then we define $(F\#G)(c)(p \leq q) = \phi_{p,q}$ as follows (we shall make a habit of dropping the subscript η and θ):
if $q = p1$, then $\phi_{p,q} = \theta(c_p)$; if $q = p0$, then $\phi_{p,q} = \eta(c_p)$;
if $p = r0$ and $q = r10$, then $\phi_{p,q} = F(\phi_{r,r1})$; if $p = r1$ and $q = r01$, then $\phi_{p,q} = G(\phi_{r,r0})$.
For $f : a \to b$ in $\mathbf{C}$ and $p \in P$, $(F\#G)(f)(p) = f_p$, where $f_{\varnothing} = f$, $f_{p0} = Ff_p$, $f_{p1} = Gf_p$.

We now check in several steps that $F\#G$ is a functor from $\mathbf{C}$ to $\mathbf{C}^{\mathbf{P}}$.

(1) Suppose $p, q, r, s \in P$; q and r cover p, and s covers q and r. Then $\phi_{q,s} \circ \phi_{p,q} = \phi_{r,s} \circ \phi_{p,r}$.

Proof. There are two kinds of cases, occurring in diagram 6.2 as triangles and parallelograms respectively.

(*Parallelograms*) Let $q = p1$ and $s = r1$; then

$$\phi_{q,s} \circ \phi_{p,q} = G\phi_{p,r} \circ \theta(c_p) = \theta(c_r) \circ \phi_{p,r} = \phi_{r,s} \circ \phi_{p,r}$$

by naturality of θ.

(*Triangles*) There are a number of symmetric cases; consider the situation in diagram 6.3.

$$\begin{tikzcd}
 & 0 \arrow[r, "\theta"] \arrow[ddr, "F\theta"'] & 01 \arrow[r, "\eta"] \arrow[ddr, "GF\theta"'] & 010 \arrow[r, "\theta"] & 0101 \\
\varnothing \arrow[ur, "\eta"] \arrow[dr, "\theta"'] & & & & \\
 & 1 \arrow[r, "\eta"'] \arrow[uur, "G\eta"] & 10 \arrow[r, "\theta"'] \arrow[uur, "FG\eta"'] & 101 \arrow[uur, "GFG\eta"'] &
\end{tikzcd} \tag{6.3}$$

First take $p = 0$, $q = 01$ (so $r = 10$ and $s = 010$). Then

$$\begin{aligned}
\phi_{r,s} \circ \phi_{p,r} &= FG\eta_c \circ F\theta_c = F(G\eta_c \circ \theta_c) = F(\theta_{c0} \circ \eta_c) && \text{since } \theta : 1_{\mathbf{C}} \to G, \\
&= F\theta_{c0} \circ \eta_{c0} && \text{since } \eta F = F\eta, \\
&= \eta_{c01} \circ \theta_{c0} && \text{since } \eta : 1_{\mathbf{C}} \to F, \\
&= \phi_{q,s} \circ \phi_{p,q} .
\end{aligned}$$

Now suppose $\phi_{q,s} \circ \phi_{p,q} = \phi_{r,s} \circ \phi_{p,r}$, with $q = p1$, $s = q0$. Then $G(\phi_{q,s} \circ \phi_{p,q}) = G(\phi_{r,s} \circ \phi_{p,r})$. But

$$G(\phi_{r,s} \circ \phi_{p,r}) = \phi_{r1,s1} \circ \phi_{p1,r1} = \phi_{r1,s1} \circ \phi_{q,r1}$$

and $G(\phi_{q,s} \circ \phi_{p,q}) = G(\eta_{c_q} \circ \theta_{c_p}) = G\eta_{c_q} \circ \theta_{c_q} = \theta_{c_s} \circ \eta_{c_q} = \phi_{s,s1} \circ \phi_{q,s}$ as $c_q = Gc_p$, $G\theta = \theta G$ and $\theta : 1_{\mathbf{C}} \to G$. ¶

We conclude by (1) that we may define $(F\#G)(c)(p < q) = \phi_{p,q}$ generally as a composition

$$\phi_{p_0,p_1} \circ \cdots \circ \phi_{p_{n-1},p_n}$$

with $p_0 = p$, $p_n = q$ en $p_k \prec p_{k+1}$ for $0 \le k < n$ $(n > 0)$; $\phi_{p,p} = 1_{c_p}$. Then

(2) $(F\#G)(c)$ is a functor from **P** to **C**.

Take $f : a \to b$ in **C**. Then

$$f_{p0} \circ (F\#G)(a)(p \le p0) = Ff_p \circ \eta_{a_p} = \eta_{b_p} \circ f_p = (F\#G)(b)(p \le p0) \circ f_p$$

by naturality of η and θ; and analogously for sequences ending in 1. Furthermore, if $p=r1$ and $q=s1$, with $r \prec s$ and

$$f_s \circ (F\#G)(a)(r{\le}s) = (F\#G)(b)(r{\le}s) \circ f_r \,,$$

(which we have just seen to hold at least for $r{=}\varnothing$ en $s{=}0$), then

$$f_q \circ (F\#G)(a)(p \le q) = (F\#G)(b)(p \le q) \circ f_p \ .$$

To see this, set $(F\#G)(a)(x{\le}y) =: \phi_{xy}$ en $(F\#G)(b)(x{\le}y) =: \psi_{xy}$. Then

$$f_q{\circ}\phi_{pq} = Gf_s{\circ}G\phi_{rs} = G(f_s{\circ}\phi_{rs}) = G(\psi_{rs}{\circ}f_r) = \psi_{pq}{\circ}f_p \ .$$

(Analogously for sequences ending in 0.) Hence

(3) $(F\#G)(f)$ is a natural transformation from $(F\#G)(a)$ to $(F\#G)(b)$.

Now it is not hard to see that $(F\#G)(f{\circ}g) = (F\#G)(f) \bullet (F\#G)(g)$, and $(F\#G)(1_c) = F\#G$. Hence

6.8 Proposition. Let $\eta : 1_{\mathbf{C}} \to F$ and $\theta : 1_{\mathbf{C}} \to G$ be uniformities in $\mathbf{C}$. Then $F\#G$ is a functor from $\mathbf{C}$ to $\mathbf{C}^{\mathbf{P}}$.

Let $\mathbf{C}$, η, F, θ and G be as above, $\mathbf{I}$ a directed ordering, $L : \mathbf{I} \to \mathbf{C}$ a functor, and $\phi : L \xrightarrow{\bullet} c$ a colimit cone. Consider $(F\#G) \circ L : \mathbf{I} \to \mathbf{C}^{\mathbf{P}}$. For every $p \in P$, let E_p be the *evaluation functor* from $\mathbf{C}^{\mathbf{P}}$ to $\mathbf{C}$, defined by $E_p(H){=}H(p)$ and $E_p(\alpha){=}\alpha_p$. We want to prove:

(4) $E_p{\circ}(F\#G) \circ L$ has colimit $(F\#G)(c)(p)$.

For this purpose we define for every p a colimit cone

$$\phi^p \colon E_p{\circ}(F\#G) \circ L \xrightarrow{\bullet} (F\#G)(c)(p).$$

For a start, $E_\varnothing{\circ}(F\#G) \circ L = L$ and $(F\#G)(c)(\varnothing) = c$; so we may take $\phi^\varnothing{=}\phi$. Now take $p{=}q0$, and suppose $\phi^q : E_q{\circ}(F\#G) \circ L \xrightarrow{\bullet} (F\#G)(c)(q)$ is a colimit cone. Since for all $i \in I$,

$$E_p((F\#G)(Li)) = (F\#G)(Li)(p) = F((F\#G)(Li)(q)),$$

we have $E_p{\circ}(F\#G) \circ L = F \circ E_q{\circ}(F\#G) \circ L$. But F is continuous, so

$$F\phi^q : E_p((F\#G)(Li)) \xrightarrow{\bullet} (F\#G)(c)(p)$$

is a colimit cone; we set $\phi^p = F\phi^q$. In this way we define ϕ^p inductively, for all $p \in P$; and we see

(5) for all $i \in I$, $\phi^p_i = (F\#G)(\phi_i)(p)$.

The existence of the parametrized colimit cone ϕ^p implies by A.5 (in the appendix) the existence of a unique functor $K : \mathbf{P} \to \mathbf{C}$ with the property that

$$\Phi : i \mapsto \lambda p.\, \phi_i^p$$

is a natural transformation from $(F\#G)\circ L$ to K, and moreover that Φ is a colimit cone. But $(F\#G)(c)(p)$ is the codomain of $\Phi(i)(p)$, so on objects $(F\#G)(c)$ works exactly as K; and since $\mathbf{P}$ is an ordering, the arrows cannot make a difference. We conclude:

(6) Φ is a colimit cone from $(F\#G)\circ L$ to $(F\#G)(c)$.

We observe that it is apparent from (5) that $\Phi = (F\#G)\circ\phi$. We have proved:

6.9 Proposition. If $\eta : 1_{\mathbf{C}} \to F$ and $\theta : 1_{\mathbf{C}} \to G$ are uniformities in a category $\mathbf{C}$, then $F\#_{\eta,\theta}G$ is a continuous functor.

6.10 Definition. Let $\eta_i : 1_{\mathbf{C}} \to F_i$ and $\theta_i : 1_{\mathbf{C}} \to G_i$ (i=0,1) be uniformities in a category $\mathbf{C}$, and $\alpha : \eta_0 \to \eta_1$ and $\beta : \theta_0 \to \theta_1$. For objects c of $\mathbf{C}$ we define $(\alpha\#\beta)_c = \langle(\alpha\#\beta)_{cp} | p\in P\rangle$ as follows:
$(\alpha\#\beta)_{c\varnothing} = 1_c$;
$(\alpha\#\beta)_{c,q0} = F_1((\alpha\#\beta)_{cq})\circ\alpha((F_0\#G_0)(c)(q))$;
$(\alpha\#\beta)_{c,q1} = G_1((\alpha\#\beta)_{cq})\circ\beta((F_0\#G_0)(c)(q))$.

We now show that $(\alpha\#\beta)_{cp}$ is natural in p and c.
Naturality in p: We consider two representative cases.
— $(\alpha\#\beta)_{c,q0}\circ\eta_0((F_0\#G_0)(c)(q)) =$

$$\begin{aligned} &F_1((\alpha\#\beta)_{cq})\circ\alpha((F_0\#G_0)(c)(q))\circ\eta_0((F_0\#G_0)(c)(q)) \\ &= F_1((\alpha\#\beta)_{cq})\circ\eta_1((F_0\#G_0)(c)(q)) \\ &= \eta_1((F_1\#G_1)(c)(q))\circ(\alpha\#\beta)_{cq}\,, \end{aligned}$$

since $\alpha\eta_0 = \eta_1$ and η_1 is natural.
— Take $i\in\{0,1\}$. Suppose

$$(\alpha\#\beta)_{c,ip}\circ(F_0\#G_0)(c)(p\le ip) = (F_1\#G_1)(c)(p\le ip)\circ(\alpha\#\beta)_{cp}\,.$$

Then since α is natural,

$(\alpha\#\beta)_{c,ip0}\circ(F_0\#G_0)(c)(p0\le ip0) =$

$$\begin{aligned} &F_1((\alpha\#\beta)_{c,ip})\circ\alpha((F_0\#G_0)(c)(ip))\circ F_0((F_0\#G_0)(c)(p\le ip)) \\ &= \alpha((F_1\#G_1)(c)(ip))\circ F_0((\alpha\#\beta)_{c,ip})\circ F_0((F_0\#G_0)(c)(p\le ip)) \\ &= \alpha((F_1\#G_1)(c)(ip))\circ F_0((\alpha\#\beta)_{c,ip}\circ(F_0\#G_0)(c)(p\le ip)) \\ &= F_1((\alpha\#\beta)_{c,ip}\circ(F_0\#G_0)(c)(p\le ip))\circ\alpha((F_0\#G_0)(c)(p)) \\ &= F_1((F_1\#G_1)(c)(p\le ip)\circ(\alpha\#\beta)_{cp})\circ\alpha((F_0\#G_0)(c)(p)) \end{aligned}$$

$$= F_1((F_1\#G_1)(c)(p\leq ip))\circ F_1((\alpha\#\beta)_{cp})\circ\alpha((F_0\#G_0)(c)(p))$$

$$= (F_1\#G_1)(c)(p0\leq ip0)\circ(\alpha\#\beta)_{c,p0}.$$

By composing such cases, we see that $(\alpha\#\beta)_c$ is natural.
Naturality in c: suppose $g: a\to b$ in $\mathbf{C}$. We must show that for every $p\in P$,

$$(*)\qquad (\alpha\#\beta)_{bp}\circ(F_0\#G_0)(g)(p) = (F_1\#G_1)(g)(p)\circ(\alpha\#\beta)_{ap}.$$

For $p=\varnothing$, this is trivial. For the rest we consider one representative case. Assume that (*) holds for a certain $p\in P$; say that $p0\in P$ as well. Then

$$(\alpha\#\beta)_{b,p0}\circ(F_0\#G_0)(g)(p0) =$$

$$F_1((\alpha\#\beta)_{bp})\circ\alpha((F_0\#G_0)(b)(p))\circ F_0((F_0\#G_0)(g)(p))$$

$$= F_1((\alpha\#\beta)_{bp})\circ F_1((F_0\#G_0)(g)(p))\circ\alpha((F_0\#G_0)(a)(p))$$

$$= F_1((\alpha\#\beta)_{bp}\circ(F_0\#G_0)(g)(p))\circ\alpha((F_0\#G_0)(a)(p))$$

$$= F_1((F_1\#G_1)(g)(p)\circ(\alpha\#\beta)_{ap})\circ\alpha((F_0\#G_0)(a)(p))$$

$$= F_1((F_1\#G_1)(g)(p))\circ F_1((\alpha\#\beta)_{ap})\circ\alpha((F_0\#G_0)(a)(p))$$

$$= (F_1\#G_1)(g)(p0)\circ(\alpha\#\beta)_{a,p0}.$$

Let α and β be as above, and $\gamma:\eta_1\to\eta_2$, $\delta:\theta_1\to\theta_2$ in $\mathbf{Un}(\mathbf{C})$. We are going to show that $(\gamma\#\delta)\bullet(\alpha\#\beta) = (\gamma\bullet\alpha)\#(\delta\bullet\beta)$, i.e.

$$(*)\qquad (\gamma\#\delta)_{cp}\circ(\alpha\#\beta)_{cp} = ((\gamma\bullet\alpha)\#(\delta\bullet\beta))_{cp}$$

for each c in $\mathbf{C}$ and $p\in P$. For $p=\varnothing$ this is trivial. Now suppose (*) holds for a certain $p\in P$; and say that $p0\in P$. Then by naturality of γ,

$$(\gamma\#\delta)_{c,p0}\circ(\alpha\#\beta))_{c,p0} =$$

$$F_2((\gamma\#\delta)_{cp})\circ\gamma((F_1\#G_1)(c)(p))\circ F_1((\alpha\#\beta)_{cp})\circ\alpha((F_0\#G_0)(c)(p))$$

$$= F_2((\gamma\#\delta)_{cp})\circ F_2((\alpha\#\beta)_{cp})\circ\gamma((F_0\#G_0)(c)(p))\circ\alpha((F_0\#G_0)(c)(p))$$

$$= F_2(((\gamma\bullet\alpha)\#(\delta\bullet\beta))_{cp})\circ(\gamma\bullet\alpha)((F_0\#G_0)(c)(p))$$

$$= ((\gamma\bullet\alpha)\#(\delta\bullet\beta))_{c,p0}.$$

We conclude:

6.11 Theorem. # is a functor from $\mathbf{Un}(\mathbf{C})\times\mathbf{Un}(\mathbf{C})$ to $(\mathbf{C}^{\mathbf{P}})^{\mathbf{C}}$.

Assume that the functors $(F\#G)(c):\mathbf{P}\to\mathbf{C}$ have colimits. We denote the colimit of $(F\#G)(c)$ by $(F+G)(c)$ (in which, strictly speaking, + should be indexed in the same way as #); and for $f: a\to b$ in $\mathbf{C}$, the unique

$$g:(F+G)(a)\to(F+G)(b)$$

such that for colimit cones

$$\phi : (F\#G)(a) \xrightarrow{\bullet} (F+G)(a) \text{ and } \psi : (F\#G)(b) \xrightarrow{\bullet} (F+G)(b),$$

$g\bullet\phi = \psi\bullet(F\#G)(f)$, by $(F+G)(f)$. Then evidently $F+G$ is a functor from $\mathbf{C}\times\mathbf{C}$ to $\mathbf{C}$. Furthermore, if $\alpha : \eta_0 \to \eta_1$ and $\beta : \theta_0 \to \theta_1$, c is an object in $\mathbf{C}$, and we have colimit cones

$$\phi : (F_0\#G_0)(c) \xrightarrow{\bullet} (F_0+G_0)(c) \text{ and } \psi : (F_1\#G_1)(c) \xrightarrow{\bullet} (F_1+G_1)(c),$$

we define $(\alpha+\beta)_c$ to be the unique $h : (F_0+G_0)(c) \to (F_1+G_1)(c)$ such that

$$h\bullet\phi = \psi\bullet(\alpha\#\beta)_c.$$

We conclude:

6.12 Lemma. The assignment

$$\langle\alpha,\beta\rangle \mapsto \alpha+\beta$$

is a functor from $\mathbf{Un}(\mathbf{C})\times\mathbf{Un}(\mathbf{C})$ to $\mathbf{C}^{\mathbf{C}}$.

Let $\eta : 1_{\mathbf{C}} \to F$ and $\theta : 1_{\mathbf{C}} \to G$ be uniformities in $\mathbf{C}$. Suppose c is an object in $\mathbf{C}$, and ϕ is the colimit cone from $(F\#G)(c)$ to $(F+G)(c)$. As was noted above (6.3.1), $1_{\mathbf{C}}$ is a uniformity and (6.6.1) $\eta : 1_{\mathbf{C}} \to \eta$ and $\theta : 1_{\mathbf{C}} \to \theta$. It is easy to see that $(1_{\mathbf{C}}\#1_{\mathbf{C}})(c)(f) = 1_c$ for each $c\in C$ and $f\in \mathbf{P}$, and hence $1_{\mathbf{C}} + 1_{\mathbf{C}} = 1_{\mathbf{C}}$. As a consequence we have

(1) $\quad \eta+\theta : 1_{\mathbf{C}} \to F+G$,

and $(\eta+\theta)_c$ is $(\phi\bullet(\eta\#\theta)_c)_\varnothing = \phi_\varnothing$. Now let ψ be the colimit cone

$$(F\#G)((F+G)(c)) \xrightarrow{\bullet} (F+G)^2(c).$$

Then $(\eta\#\theta)((F+G)(c))(\varnothing)\circ\phi_\varnothing = \phi_\varnothing = (\eta+\theta)_c = (F\#G)((\eta+\theta)_c)(\varnothing)$. If $p=q0$, then

$$\phi_q = \phi_p\circ\eta_{(F\#G)(c)(q)},$$

so $F\phi_q = F\phi_p\circ F\eta_{(F\#G)(c)(q)} = F\phi_p\circ\eta_{F((F\#G)(c)(q))} = \eta_{(F+G)(c)}\circ\phi_p$ since η is a uniformity from $1_{\mathbf{C}}$ to F. So if we assume that

$$(\eta\#\theta)((F+G)(c))(q)\circ\phi_q = (F\#G)((\eta+\theta)_c)(q),$$

we get

$$\begin{aligned}(\eta\#\theta)((F+G)(c))(p)\circ\phi_p &= F((\eta\#\theta)((F+G)(c))(q))\circ\eta((F+G)(c))\circ\phi_p \\ &= F((\eta\#\theta)((F+G)(c))(q))\circ F\phi_q \\ &= F((\eta\#\theta)((F+G)(c))(q)\circ\phi_q)\end{aligned}$$

$$= F((F\#G)((\eta+\theta)_c)(q))$$

$$= (F\#G)((\eta+\theta)_c)(p).$$

So $(\eta\#\theta)((F+G)(c))\bullet\phi = (F\#G)((\eta+\theta)_c)$, and therefore

$$\psi\bullet(F\#G)((\eta+\theta)_c) = \psi\bullet(\eta\#\theta)((F+G)(c))\bullet\phi.$$

From the trivial nature of $1_{\mathbf{C}}\#1_{\mathbf{C}}$ we see that $(\eta+\theta)((F+G)(c))$ is defined as

$$\psi\bullet(\eta\#\theta)(F+G)(c)\,;$$

so $\psi\bullet(F\#G)((\eta+\theta)_c) = (\eta+\theta)((F+G)(c))\bullet\phi$. By the definition of $F+G$ we have $(F+G)((\eta+\theta)_c) = (\eta+\theta)((F+G)(c))$, and hence

(2) $\quad (F+G)\circ(\eta+\theta) = (\eta+\theta)\circ(F+G).$

Let $\mathbf{I}$ be a directed ordering, $L:\mathbf{I}\to\mathbf{C}$ a functor, and $\phi: L\stackrel{\bullet}{\to} c$ a colimit cone. Let $\psi:(F+G)L\stackrel{\bullet}{\to} d$ be any cocone. By 6.9, $(F\#G)(c)$ is a colimit of $(F\#G)L$, with colimit cone $\Phi = (F\#G)\circ\phi$. For $i\in I$, let

$$X_i : (F\#G)(Li) \stackrel{\bullet}{\to} (F+G)(Li)$$

be the colimit cone, then

$$X := \lambda p\in P.(\lambda i\in I.X_{ip}) : (F\#G)L \stackrel{\bullet}{\to} (F+G)L$$

is a colimit cone in $\mathbf{C}^{\mathbf{I}}$ by proposition A.5. Now we have $\psi\bullet X:(F\#G)L\stackrel{\bullet}{\to} d$, hence there exists a unique $\chi:(F\#G)(c)\stackrel{\bullet}{\to} d$ with the property that $\psi\bullet X = \chi\bullet\Phi$. Let ω be the colimit cone from $(F\#G)(c)$ to $(F+G)(c)$. There is a unique $f:(F+G)(c)\stackrel{\bullet}{\to} d$ with the property that $f\bullet\omega=\chi$. By the definition of $F+G$ we have, for all $i\in I$ and $p\in P$,

$$(\omega\bullet\Phi)(i)(p) = (\omega\bullet(F\#G)(\phi_i))(p) = \omega_p\circ(F\#G)(\phi_i)(p) = (F+G)(\phi_i)\circ X_{ip}$$
$$= ((F+G)(\phi_i)\bullet X_i)(p) = (((F+G)\circ\phi)\bullet X)(i)(p)\,.$$

$$\begin{tikzcd}
(F\#G)L \arrow[r, "\Phi"] \arrow[d, "X"] & (F\#G)(c) \arrow[d, "\omega"] \arrow[dr, "\chi"] & \\
(F+G)L \arrow[r, "(F+G)\phi"'] & (F+G)(c) \arrow[r, "f"'] & d
\end{tikzcd}$$

(6.4)

So diagram 6.4 commutes, hence $f\bullet(F+G)\phi\bullet X = \chi\bullet\Phi = \psi\bullet X$; unique factorization over the colimit cone X then implies that $f\bullet(F+G)\phi = \psi$. The last identity determines f uniquely: if $g\bullet(F+G)\phi\bullet X = \chi\bullet\Phi$, then $g\bullet\omega\bullet\Phi = \chi\bullet\Phi$, so $g\bullet\omega=\chi$, so $g=f$. We have proved:

(3) $F+G$ is continuous.

If $\alpha: \eta_0 \to \eta_1$ and $\beta: \theta_0 \to \theta_1$, that is, $\alpha \bullet \eta_0 = \eta_1$ en $\beta \bullet \theta_0 = \theta_1$, then by lemma 6.12, $(\alpha+\beta) \bullet (\eta_0+\theta_0) = \eta_1+\theta_1$, i.e. $\alpha+\beta: \eta_0+\theta_0 \to \eta_1+\theta_1$ in $\mathbf{Un}(\mathbf{C})$. So by lemma 6.12 and (1), (2) and (3):

6.13 Theorem. Let $\mathbf{C}$ be a category. The operation + is a functor

$$\mathbf{Un}(\mathbf{C}) \times \mathbf{Un}(\mathbf{C}) \to \mathbf{Un}(\mathbf{C}) .$$

If we change the order of the arguments of $F\#G$, writing $(F\#G)(p)(c)$ instead of $(F\#G)(c)(p)$, we turn $F\#G$ into a functor from $\mathbf{P}$ to $\mathbf{C}^{\mathbf{C}}$, and $F+G$ becomes a colimit of $F\#G$ in $\mathbf{C}^{\mathbf{C}}$. We shall take this view now.

6.14 Lemma. Let η be a uniformity in a category $\mathbf{C}$; then $\eta+\eta \cong \eta$.

Proof. Suppose $\eta: 1_{\mathbf{C}} \to F$; then $F \cong FF$, so for all $p > \varnothing$ in $\mathbf{P}$, $(F\#F)(p) \cong F$. So if $\phi: F\#F \xrightarrow{\bullet} F+F$ is the colimit cone, ϕ_p is an isomorphism, for all $p > \varnothing$. ¶

6.15 Corollary. Let η be as above; then $\eta \circ (F+F)$: $F+F \cong F \circ (F+F)$.

Proof. We have $F+F \cong F \cong FF \cong F \circ (F+F)$; the isomorphism (given colimit cone $\phi: F\#F \xrightarrow{\bullet} F+F$) is

$$F\phi_0 \bullet \eta F \bullet (\phi_0)^{-1} = \eta \circ (F+F) \bullet \phi_0 \bullet (\phi_0)^{-1} = \eta \circ (F+F) . \qquad ¶$$

6.16 Theorem. Let η and θ be uniformities in a category $\mathbf{C}$. Then $\eta+\theta$ is a coproduct of η and θ.

Proof. Take $\eta: 1_{\mathbf{C}} \to F$, $\theta: 1_{\mathbf{C}} \to G$ and $\zeta: 1_{\mathbf{C}} \to H$; $\alpha: \eta \to \zeta$ and $\beta: \theta \to \zeta$; let ϕ be the colimit cone from $F\#G$ to $F+G$, ψ that from $H\#H$ to $H+H$. We shall prove that there exists exactly one $\kappa: \eta+\theta \to \zeta$ with the property that $\kappa \bullet \phi_0 = \alpha$ en $\kappa \bullet \phi_1 = \beta$; i.e. that $\eta+\theta$ is a coproduct of η and θ, with injections ϕ_0 and ϕ_1.

By the proof of lemma 6.14, ψ_0 is invertible; and by definition of $\alpha+\beta$, $(\alpha+\beta) \bullet \phi_0 = \psi_0 \bullet \alpha$. So

(1) $(\psi_0)^{-1} \bullet (\alpha+\beta) \bullet \phi_0 = \alpha$.

Since $\psi_1 = \psi_0$, likewise

(2) $(\psi_0)^{-1} \bullet (\alpha+\beta) \bullet \phi_1 = \beta$.

So $(\psi_0)^{-1} \bullet (\alpha+\beta)$ meets the requirements on κ.

Unicity will be considerably harder. Suppose $\lambda: \eta+\theta \to \zeta$, with $\lambda\phi_0 = \alpha$ and $\lambda\phi_1 = \beta$. We shall prove that $\psi_0 \bullet \lambda = \alpha+\beta$ (and hence

$$\lambda = (\psi_0)^{-1}\bullet(\alpha+\beta)\ ).$$

By the definition of $\alpha+\beta$ it will suffice to prove that for every $p \in P$,

$$\psi_0\bullet\lambda\bullet\phi_p = \psi_p\bullet(\alpha\#\beta)_p\ .$$

We have $\psi_0\bullet\lambda\bullet\phi_0 = \psi_0\bullet\alpha = \psi_0\bullet(\alpha\#\beta)_0$, analogously $\psi_0\bullet\lambda\bullet\phi_1 = \psi_1\bullet(\alpha\#\beta)_1$, and therefore

$$\begin{aligned}\psi_0\bullet\lambda\bullet\phi_\varnothing &= \psi_0\bullet\lambda\bullet\phi_0\bullet\eta = \psi_0\bullet(\alpha\#\beta)_0\bullet\eta = \psi_0\bullet\alpha\bullet\eta = \psi_0\bullet\zeta = \psi_\varnothing \\ &= \psi_\varnothing\bullet(\alpha\#\beta)_\varnothing .\end{aligned}$$

Now suppose $\psi_0\bullet\lambda\bullet\phi_p = \psi_p\bullet(\alpha\#\beta)_p$ (induction hypothesis); we want to prove that $\psi_0\bullet\lambda\bullet\phi_{p0} = \psi_{p0}\bullet(\alpha\#\beta)_p$. If we succeed, we are done, by induction and symmetry. Since $\zeta\circ(H+H)$ is invertible (corollary 6.15), it is sufficient to show that

$$(\zeta\circ(H+H))\bullet\psi_0\bullet\lambda\bullet\phi_{p0} = (\zeta\circ(H+H))\bullet\psi_{p0}\bullet(\alpha\#\beta)_{p0}\,.$$

By naturality of $\zeta : 1 \to H$ and since $\psi_0 : H \to H+H$,

(3) $\quad (\zeta\circ(H+H))\bullet\psi_0 = H\psi_0\bullet\zeta H.$

Since moreover $\lambda : F+G \to H$, we have

(4) $\quad \zeta H\bullet\lambda = H\lambda\bullet(\zeta\circ(F+G))$.

Observe that

(5) $F\phi_p = F\phi_{p0}\bullet(F\eta\circ(F\#G)(p)) = F\phi_{p0}\bullet(\eta\circ(F\#G)(p0)) = (\eta\circ(F+G))\bullet\phi_{p0}\,.$

Since $\zeta{=}\alpha\bullet\eta$, and hence $\zeta\circ(F+G) = (\alpha\circ(F+G))\bullet(\eta\circ(F+G))$; and since $\alpha : F \to H$ and $(F\#G)(p0) = (F\circ(F\#G))(p)$ — (5) implies

(6) $\quad (\zeta\circ(F+G))\bullet\phi_{p0} = H\phi_p\bullet(\alpha\circ(F\#G)(p)).$

From (3), (4) and (6) we get

(7) $\quad (\zeta\circ(H+H))\bullet\psi_0\bullet\lambda\bullet\phi_{p0} = (H\circ(\psi_0\bullet\lambda\bullet\phi_p))\bullet(\alpha\circ(F\#G)(p)).$

The type of ψ_{p0} is $(H\#H)(p0) \to H+H$. By naturality of ζ,

(8) $\quad (\zeta\circ(H+H))\bullet\psi_{p0} = H\psi_{p0}\bullet(\zeta\circ(H\#H)(p0)),$

and since $\alpha\#\beta : F\#G \to H\#H$,

(9) $\quad (\zeta\circ(H\#H)(p0))\bullet(\alpha\#\beta)(p0) = (H\circ(\alpha\#\beta)(p0))\bullet(\zeta\circ(F\#G)(p0))$.

Using the definition of $(\alpha\#\beta)(p0)$ and once more the naturality of ζ, we see

(10) $\quad (H\circ(\alpha\#\beta)(p0))\bullet(\zeta\circ(F\#G)(p0)) =$

$$(H^2\circ(\alpha\#\beta)(p))\bullet(\zeta H\circ(F\#G)(p))\bullet(\alpha\circ(F\#G)(p))$$

and considering that $\zeta H = H\zeta$,

(11) $(H^2\circ(\alpha\#\beta)(p))\bullet(\zeta H\circ(F\#G)(p)) = (H\zeta\circ(H\#H)(p))\bullet(H\circ(\alpha\#\beta)(p))$.

By (8)-(11), since $\psi_{p0}\bullet(\zeta\circ(H\#H)(p)) = \psi_p$,

(12) $(\zeta\circ(H+H))\bullet\psi_{p0}\bullet(\alpha\#\beta)(p0) = H\circ(\psi_p\bullet(\alpha\#\beta)(p))\bullet(\alpha\circ(F\#G)(p))$;

now by (7) and the induction hypothesis we are done. ¶

Given a coproduct diagram

$$\begin{tikzcd}
a \arrow[r, "\kappa"] \arrow[dr, "f"'] & a+b & b \arrow[l, "\kappa'"'] \arrow[dl, "g"] \\
& c &
\end{tikzcd} \tag{6.4}$$

the unique $h : a+b \to c$ such that $h\kappa = f$ and $h\kappa' = g$ is denoted by $[f, g]$; and if c happens to be a coproduct as well, say with injections λ and λ', $[\lambda k, \lambda' l]$ is abbreviated to $k+l$. Our notation agrees with this custom: if $\alpha : \eta \to \eta'$ en $\beta : \theta \to \theta'$, and ϕ is the colimit cone associated with $\eta + \theta$, ψ the colimit cone associated with $\eta' + \theta'$, then

$$(\alpha+\beta)\circ\phi_i = \psi_i\circ(\alpha+\beta) \qquad (i=0,1)$$

by definition, so $\alpha+\beta = [\psi_0\circ\alpha,\ \psi_1\circ\beta]$.

7. Algebraic uniformities

A uniformity η_K is tied to a particular signature. We shall need this connection in the form presented by the following definition.

7.1 Definition. A uniformity $\eta : 1 \to F$ in **Alg** is *algebraic* if a signature σ exists such that for each $\mathbf{A} \in$ **Alg**, $\Sigma F\mathbf{A} \subseteq \Sigma\mathbf{A} \cup \sigma$, and for any two D-bijections ϕ and ψ with $\phi|\sigma = \psi|\sigma$, $\phi\cdot\eta \cong \psi\cdot\eta$.

The full subcategory of **Un**(**Alg**) with the algebraic uniformities for objects I call **AUn**.

Given a uniformity η in **Alg**, let us call a signature σ an η-*basis* (or *basis for* η) if $\phi\cdot\eta \cong \psi\cdot\eta$ for all D-bijections ϕ and ψ that agree on σ. We shall see that the class of η-bases is closed under finite intersection.

7.2 Lemma. Let σ, τ and υ be signatures and ψ a D-bijection. Then a D-bijection χ exists such that $\chi|\sigma = \psi|\sigma$ and for all $\mathbf{d} \in \tau$, $\chi(\mathbf{d}) \in \upsilon$ only if $\mathbf{d} \in \sigma$.

Proof. Let $\mathrm{N} = \{\mathrm{n}_i \mid i \in \mathbb{N}\}$, as in §2. Pick $M \in \mathbb{N}$ greater than all indices of names in υ. Define:
$\chi_0(\mathbf{d}) = \psi(\mathbf{d})$ for $\mathbf{d} \in \sigma$;

$\chi_0(n_i) = n_{M+i}$ for $n_i \in \tau-\sigma$;
$\chi_0(n_i, m_0, \ldots, m_k) = \langle n_{M+i}, \chi_0(m_0), \ldots, \chi_0(m_k)\rangle$ for $\langle n_i, m_0, \ldots, m_k\rangle \in \tau-\sigma$.
Then χ_0 is an injective morphism from $\sigma\cup\tau$ to $\chi_0(\sigma\cup\tau)$; and if $\mathbf{d}\notin\sigma$, then $\chi_0(\mathbf{d})\notin \upsilon$. By lemma 2.2 there exists a D-bijection $\chi\supseteq\chi_0$; χ has the required properties. ¶

7.3 Theorem. If σ and τ are η-bases, then so is $\sigma\cap\tau$.

Proof. Let ϕ and ψ be D-bijections that agree on $\sigma\cap\tau$. Construct a D-bijection χ with the properties

$$\chi|\sigma = \psi|\sigma \text{ and for all } \mathbf{d}\in\tau,\ \chi(\mathbf{d})\in \phi\cdot\sigma \text{ only if } \mathbf{d}\in\sigma.$$

(use the lemma with $\upsilon=\phi\cdot\sigma$.) Since σ is an η-basis, we must have $\psi\cdot\eta \cong \chi\cdot\eta$. Observe that χ agrees with ϕ on $\sigma\cap\tau$. Now let

$$\omega_0 = \{\langle \mathbf{d}, \phi(\mathbf{d})\rangle | \mathbf{d}\in\sigma\} \cup \{\langle \mathbf{d}, \chi(\mathbf{d})\rangle | \mathbf{d}\in\tau\}.$$

Then ω_0 is a signature morphism, injective by the choice of χ. Extend ω_0 to a D-bijection ω, by lemma 2.2. Then $\omega|\sigma = \phi|\sigma$ and $\omega|\tau = \chi|\tau$, so $\phi\cdot\eta \cong \omega\cdot\eta \cong \chi\cdot\eta$. ¶

Hence the class of η-bases is closed under finite intersections. If η has a basis σ, then by this theorem η has a *least* basis, for then the class Σ of all η-bases may be written as $\sigma\cap\bigcap(\tau_{\mathbf{d}} | \mathbf{d}\in\sigma-\bigcap\Sigma)$, for a finite family of η-bases with $\mathbf{d}\notin\tau_{\mathbf{d}}$. Since obviously if $\Sigma F\mathbf{A}\subseteq\Sigma\mathbf{A}\cup\sigma$ and $\Sigma F\mathbf{A}\subseteq\Sigma\mathbf{A}\cup\tau$, we must have $\Sigma F\mathbf{A}\subseteq\Sigma\mathbf{A}\cup(\sigma\cap\tau)$ as well, an algebraic uniformity η has a *least* signature σ satisfying the conditions of definition 7.1, which may well be called *the* signature of the uniformity, and will be denoted by $\Sigma(\eta)$.

7.4 Lemma. Let ϕ and ψ be D-bijections, and η a uniformity in **Alg**. Then
(i) $(\phi\psi)^* = \psi^*\circ\phi^*$;
(ii) $(\phi\psi)\cdot\eta = \phi\cdot(\psi\cdot\eta)$.

Proof. (i) Let **A** be an algebra; then

$$(\phi\psi)^*(\mathbf{A}) = \mathbf{A}\circ\phi\psi = (\mathbf{A}\circ\phi)\circ\psi = \psi^*(\phi^*(\mathbf{A})).$$

(ii) We calculate, using (i):

$$\begin{aligned}(\phi\psi)\cdot\eta &= ((\phi\psi)^{-1})^*\circ\eta\circ(\phi\psi)^* = (\psi^{-1}\circ\phi^{-1})^*\circ\eta\circ(\phi\psi)^*\\ &= (\phi^{-1})^*\circ(\psi^{-1})^*\circ\eta\circ\psi^*\circ\phi^* = \phi\cdot(\psi\cdot\eta).\end{aligned}$$ ¶

7.5 Lemma. Let ϕ be a D-bijection. If η is an algebraic uniformity, then so is $\phi\cdot\eta$; and $\Sigma(\phi\cdot\eta) = \phi\cdot\Sigma(\eta)$.

Proof. We noted in 6.3.4 that $\phi\cdot\eta$ is a uniformity. If $\Sigma F\mathbf{A} \subseteq \Sigma\mathbf{A} \cup \sigma$ for every algebra $\mathbf{A}$, then $\Sigma(F(\mathbf{A}\circ\phi)\circ\phi^{-1}) \subseteq \Sigma\mathbf{A} \cup \phi\cdot\sigma$ by proposition 5.4. Now let ψ and χ be D-bijections such that $\chi|\,\phi\cdot\Sigma(\eta) = \psi|\,\phi\cdot\Sigma(\eta)$. Then $\chi\phi|\Sigma(\eta) = \psi\phi|\Sigma(\eta)$, so $\chi\phi\cdot\eta \cong \psi\phi\cdot\eta$, and by 7.4(ii), $\chi\cdot(\phi\cdot\eta) \cong \psi\cdot(\phi\cdot\eta)$. ¶

Let ω be a D-bijection. By proposition 6.4, the assignment $\alpha \mapsto \omega\cdot\alpha$ is an automorphism of **AUn**; consequently $\omega\cdot(\eta+\theta) \cong \omega\cdot\eta + \omega\cdot\theta$. By the construction of $\eta+\theta$ and the elementary properties of coproducts we have:

7.6 Lemma. If uniformities η and θ are algebraic, then so is $\eta+\theta$, and $\Sigma(\eta+\theta) = \Sigma(\eta)\cup\Sigma(\theta)$.

We now study signature change and addition for the basic uniformities $\eta_{\boldsymbol{K}}$. We begin by recalling a definition from [BHK].

7.7 Definition. Let $\boldsymbol{K}$ be a class of σ-algebras, and $\boldsymbol{L}$ a class of τ-algebras. Then $\boldsymbol{K}+\boldsymbol{L}$ is the class of all $\sigma\cup\tau$-algebras $\mathbf{A}$ such that $\mathbf{A}|\sigma\in\boldsymbol{K}$ and $\mathbf{A}|\tau\in\boldsymbol{L}$.

The operation + is a semilattice operation on the classes of similar algebras; en *Mod* is a homomorphism of semilattices. I write $\boldsymbol{K}\leq\boldsymbol{L}$ when $\boldsymbol{K}+\boldsymbol{L}=\boldsymbol{L}$. Clearly, $\boldsymbol{K}\leq\boldsymbol{L}$ if and only if $\boldsymbol{K}\supseteq\Sigma\boldsymbol{K}\,\square\,\boldsymbol{L}$.

7.8 Lemma. Let $\boldsymbol{K}$ and $\boldsymbol{L}$ be classes of similar algebras. If $\boldsymbol{K}\leq\boldsymbol{L}$, then $\boldsymbol{G}(\boldsymbol{K})\supseteq\boldsymbol{G}(\boldsymbol{L})$.

Proof. Let $\Sigma\boldsymbol{K}=\sigma$ and $\Sigma\boldsymbol{L}=\tau$; then $\sigma\subseteq\tau$, which implies that τ-generated algebras are σ-generated. If $\mathbf{A}\in\boldsymbol{G}(\boldsymbol{L})$, there exists an algebra $\mathbf{B}\in\boldsymbol{L}$ such that $\tau\,\square\,\mathbf{A} \leq \Sigma\mathbf{A}\,\square\,\mathbf{B}$. Then $\mathbf{B}|\sigma\in\boldsymbol{K}$, and

$$\sigma\,\square\,\mathbf{A} = \sigma\,\square\,(\tau\,\square\,\mathbf{A}) \leq \sigma\,\square\,(\Sigma\mathbf{A}\,\square\,\mathbf{B}) = \Sigma\mathbf{A}\,\square\,(\mathbf{B}|\sigma),$$

that is to say, $\mathbf{A}\in\boldsymbol{G}(\boldsymbol{K})$. ¶

7.9 Proposition. Let $\boldsymbol{K}$ and $\boldsymbol{L}$ be varieties. If $\boldsymbol{K}\leq\boldsymbol{L}$, then there exists a morphism from $\eta_{\boldsymbol{K}}$ to $\eta_{\boldsymbol{L}}$.

Proof. Let $\mathbf{A}$ be an algebra; then $F_{\boldsymbol{L}}(\mathbf{A})$ is generated over $\boldsymbol{K}$, by lemma 8. Let $\alpha_{\mathbf{A}}$, for $\mathbf{A}\in\mathbf{Alg}$, be the unique homomorphism ϕ from $F_{\boldsymbol{K}}(\mathbf{A})$ to $F_{\boldsymbol{L}}(\mathbf{A})$ determined by the equation $\phi\circ\eta_{\boldsymbol{K},\mathbf{A}}=\eta_{\boldsymbol{L},\mathbf{A}}$. We need only prove that α is a natural transformation from $F_{\boldsymbol{K}}$ to $F_{\boldsymbol{L}}$. Let $f\colon\mathbf{A}\to\mathbf{B}$ be a homomorphism.

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "\eta_{K,\mathbf{A}}"] \arrow[d, "f"'] & F_K\mathbf{A} \arrow[r, "\alpha_{\mathbf{A}}"] \arrow[d, "F_K f"] & F_L\mathbf{A} \arrow[d, "F_L f"] \\
\mathbf{B} \arrow[r, "\eta_{K,\mathbf{B}}"'] & F_K\mathbf{B} \arrow[r, "\alpha_{\mathbf{B}}"'] & F_L\mathbf{B}
\end{tikzcd} \tag{7.1}$$

Then

$$F_L(f)\circ\alpha_{\mathbf{A}}\circ\eta_{K,\mathbf{A}} = F_L(f)\circ\eta_{L,\mathbf{A}} = \eta_{L,\mathbf{B}}\circ f = \alpha_{\mathbf{B}}\circ\eta_{K,\mathbf{B}}\circ f = \alpha_{\mathbf{B}}\circ F_K(f)\circ\eta_{K,\mathbf{A}},$$

by the definition of α and by naturality of η_L and η_K. The universal property of $\eta_{K,\mathbf{A}}$ now guarantees that $F_L(f)\circ\alpha_{\mathbf{A}} = \alpha_{\mathbf{B}}\circ F_K(f)$, which is what we set out to prove. ¶

7.10 Proposition. Let σ and τ be signatures with $S\sigma \subseteq S\tau$; $\boldsymbol{K}$ a σ-variety and $\boldsymbol{L}$ a τ-variety. If there exists a morphism $\eta_K \to \eta_L$, then $\boldsymbol{K} \leq \boldsymbol{L}$.

Proof. Suppose $\alpha : \eta_K \to \eta_L$, $\mathbf{B} \in \boldsymbol{L}$, and $\mathbf{A} = \sigma\square\mathbf{B}$. We must prove that $\mathbf{A} \in \boldsymbol{K}$. Observe that the existence of α implies that $F\sigma \subseteq F\tau$, so $\sigma \subseteq \tau$, and $\Sigma\mathbf{A} = \sigma$; consequently $F_K(\mathbf{A}) \in \boldsymbol{K}$ and $\eta_{K,\mathbf{A}}$ is surjective. Let $\iota : \mathbf{A} \to \mathbf{B}$ be the σ-homomorphism $\langle 1_{\mathbf{A}(S)} \mid S \in S\sigma\rangle$, and define $f : F_L(\mathbf{A}) \to \mathbf{B}$ by $\iota = f\circ\eta_{L,\mathbf{A}}$.

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "\eta"] \arrow[d, "\iota"'] \arrow[dr, "\eta"] & F_K\mathbf{A} \arrow[d, "\alpha"] \\
\mathbf{B} & F_L\mathbf{A} \arrow[l, "f"]
\end{tikzcd} \tag{7.2}$$

Since ι is a monomorphism, $\eta_{L,\mathbf{A}}$ is a monomorphism; hence likewise $\eta_{K,\mathbf{A}}$ is a monomorphism. So $\mathbf{A} \cong F_K(\mathbf{A})$, hence $\mathbf{A} \in \boldsymbol{K}$. ¶

Let $\boldsymbol{K}$ and $\boldsymbol{L}$ be varieties. By the universal property of the homomorphisms $\eta_{K,\mathbf{A}}$, there exists at most one morphism $\eta_K \to \eta_L$. Hence the full subcategory of **Un**(**Alg**) determined by the uniformities η_K for varieties $\boldsymbol{K}$ is a quasi-ordering. By proposition 7.9, the assignment $\boldsymbol{K} \mapsto \eta_K$ preserves the order. We shall now prove that it maps suprema to coproducts.

7.11 Theorem. Let $\boldsymbol{K}$ and $\boldsymbol{L}$ be varieties. Then $\eta_{K+L} \cong \eta_K + \eta_L$.

Example. When considering algebras, one might be tempted to reason as follows about the construction of $(F_K + F_L)(\mathbf{A})$: first we construct $F_K(\mathbf{A})$ as a term algebra over $\mathbf{A}$, and factor over a congruence induced by $\boldsymbol{K}$. This we repeat with $\boldsymbol{L}$ and $F_{\boldsymbol{K}}(\mathbf{A})$. When next we proceed to $F_{\boldsymbol{K}}F_{\boldsymbol{L}}F_{\boldsymbol{K}}(\mathbf{A})$, we do not get any new terms, there are only some identifications of terms that were added in the previous step; and in later steps nothing happens at all. So $(F_{\boldsymbol{K}}+F_{\boldsymbol{L}})(\mathbf{A}) \cong F_{\boldsymbol{K}}F_{\boldsymbol{L}}F_{\boldsymbol{K}}(\mathbf{A})$. — The mistake is in the supposition that no new terms arise in the construction of $F_{\boldsymbol{K}}F_{\boldsymbol{L}}F_{\boldsymbol{K}}(\mathbf{A})$. This holds if $F_{\boldsymbol{K}}(\mathbf{A}) \in \boldsymbol{K}$ and $F_{\boldsymbol{L}}(\mathbf{A}) \in \boldsymbol{L}$, but if for example

$$\mathrm{F}\Sigma\boldsymbol{K} = \{\langle \mathrm{f}_0, \mathrm{S}_0\rangle\} \cup \{\langle \mathrm{f}_n, \mathrm{T}_{n-1}, \mathrm{S}_n\rangle \mid n>0\}$$

and

$$\mathrm{F}\Sigma\boldsymbol{L} = \{\langle \mathrm{g}_0, \mathrm{T}_0\rangle\} \cup \{\langle \mathrm{g}_n, \mathrm{S}_{n-1}, \mathrm{T}_n\rangle \mid n>0\},$$

then for all $p>q$ in $\mathbf{P}$, $\Sigma(F_{\boldsymbol{K}} \# F_{\boldsymbol{L}})(\varnothing)(q) \subset \Sigma(F_{\boldsymbol{K}} \# F_{\boldsymbol{L}})(\varnothing)(p)$. If we stick to finite signatures, the example shows that the number of steps needed to arrive in a stable situation may be arbitrarily large.

Proof of Theorem 7.11. Let $F=F_{\boldsymbol{K}}$, $G=F_{\boldsymbol{L}}$, $\eta=\eta_{\boldsymbol{K}}$, $\theta=\eta_{\boldsymbol{L}}$, $H=F_{\boldsymbol{K}+\boldsymbol{L}}$ and $\zeta=\eta_{\boldsymbol{K}+\boldsymbol{L}}$; $\sigma=\Sigma\boldsymbol{K}$ and $\tau=\Sigma\boldsymbol{L}$. Take an arbitrary algebra $\mathbf{A}$. Expand the algebras $(F\#G)(\mathbf{A})(p)$ (for $p\in P$), $(F+G)(\mathbf{A})$ and $H\mathbf{A}$ with proper names for A, to $\mathbf{A}_p$, $\mathbf{A}_\infty$ and $\mathbf{B}$ respectively, as follows:
$\mathbf{A}_\varnothing = \mathbf{A}_A$;
$\mathbf{A}_{p0}(\mathbf{a}) = \eta((F\#G)(\mathbf{A})(p))(\mathbf{A}_p(\mathbf{a}))$;
$\mathbf{A}_{q1}(\mathbf{a}) = \theta((F\#G)(\mathbf{A})(q))(\mathbf{A}_q(\mathbf{a}))$;
$\mathbf{A}_\infty(\mathbf{a}) = \phi_\varnothing(a)$, where $\phi : (F\#G)(\mathbf{A}) \xrightarrow{\bullet} (F+G)(\mathbf{A})$ is the colimit cone; and $\mathbf{B}(\mathbf{a}) = \zeta_{\mathbf{A}}(a)$.

The given homomorphisms form a colimit cone for the expanded algebras as well. Each of the elements of each of the algebras $\mathbf{A}_x$ $(x\in P\cup\{\infty\})$ and $\mathbf{B}$ is an interpretation of a closed term. Put $\upsilon=\Sigma\mathbf{B}$. Each $\mathbf{A}_p$ may be seen as a quotient of a term algebra $\mathbf{T}_{\upsilon'}$, with $\upsilon'\subseteq\upsilon$. Under this aspect, the assignment $\mathbf{A}_p(\mathbf{t}) \mapsto \mathbf{B}(\mathbf{t})$ clearly is a homomorphism ψ_p from $\mathbf{A}_p$ to $\mathbf{B}$. But $\mathbf{A}_\infty$ is isomorphic to a quotient of the union of the term algebras $\mathbf{T}_{\upsilon'}$, in which $\mathbf{s}\equiv\mathbf{t}$ if and only if there is some p such that $\mathbf{A}_p \models \mathbf{s}=\mathbf{t}$; so $\mathbf{A}_\infty(\mathbf{t}) \mapsto \mathbf{B}(\mathbf{t})$ is a homomorphism ψ_∞ from $\mathbf{A}_\infty$ to $\mathbf{B}$. To prove that ψ_∞ is an isomorphism, it suffices to show that (i) for every closed term $\mathbf{t}$ over υ there exists $p\in P$ such that $\mathbf{t}$ is interpretable in $\mathbf{A}_p$, and (ii) $\Delta\mathbf{A}\cup\mathcal{E}q(\boldsymbol{K}+\boldsymbol{L}) \vdash \mathbf{s}=\mathbf{t}$ implies that there exists p such that $\mathbf{A}_p \models \mathbf{s}=\mathbf{t}$.
(i) Define T_p, for $p\in P$, as follows:
$T_\varnothing = T_{\Sigma\mathbf{A}}$;
if $p=q0$ (respectively $q1$), then T_p consists of the elements of T_υ that one obtains by substituting elements of T_q for the variables in a σ-term (respectively τ-term).

It is clear that $T_\upsilon = \bigcup_{p\in P} T_p$; *and* that all the elements of T_p are interpretable in $\mathbf{A}_p$.

(ii) If **s=t** is derivable from $\Delta\mathbf{A}\cup\mathcal{E}q(\boldsymbol{K}+\boldsymbol{L})$, then it is derivable from Γ :=

$$\Delta\mathbf{A} \cup \mathcal{E}q\boldsymbol{K} \cup \mathcal{E}q\boldsymbol{L}$$

as well; and without substitution from the set Γ^* of all closed substitution instances of Γ over υ. So it suffices to prove that for closed υ-equations α, $\Gamma^* \vdash \alpha$ implies the existence of p such that $\mathbf{A}_p \vDash \alpha$. We use induction on the length of derivations.

If $\mathbf{s}=\mathbf{t}$, take $p\in P$ such that $\mathbf{s}\in T_p$; then $\mathbf{A}_p \vDash \mathbf{s}=\mathbf{t}$. If **s=t** is a substitution instance of an equation that is valid in $\boldsymbol{K}$, then take $p,q\in P$ such that $\mathbf{s}\in T_p$ and $\mathbf{t}\in T_q$; then $\mathbf{s},\mathbf{t}\in T_{r1}$ for $r1\geq p,q$, and $\mathbf{A}_{r10}=F_{\boldsymbol{K}}(\mathbf{A}_{r1})\vDash\mathbf{s}=\mathbf{t}$. Instances of $\boldsymbol{L}$-identities are treated analogously. The cases corresponding to the rules of symmetry and transitivity are trivial. Finally, if $\mathbf{s}=\mathbf{f}\mathbf{s}_1\ldots\mathbf{s}_n$ and $\mathbf{t}=\mathbf{f}\mathbf{t}_1\ldots\mathbf{t}_n$, and $\mathbf{s}_1=\mathbf{t}_1,\ldots,$ $\mathbf{s}_n=\mathbf{t}_n$ are derivable, and $\mathbf{f}\in\sigma$ (as we may assume without loss of generality), and we have $p_1,\ldots,p_n\in P$ with $\mathbf{A}_{p_i}\vDash\mathbf{s}_i=\mathbf{t}_i$ $(1\leq i\leq n)$, then take q_1 with $\mathbf{s}\in T_{q_1}$ and q_2 with $\mathbf{t}\in T_{q_2}$, and $r1\geq p_1,\ldots,p_n,q_1,q_2$; then $\mathbf{A}_{r1}\vDash\mathbf{s}_i=\mathbf{t}_i$ $(1\leq i\leq n)$, and $\mathbf{A}_{r10}=F_{\boldsymbol{K}}(\mathbf{A}_{r1})\vDash\mathbf{s}=\mathbf{t}$. ¶

Let us call a sort S in the signature of a class $\boldsymbol{K}$ of similar algebras *isolated* with respect to $\boldsymbol{K}$ if $\Sigma\boldsymbol{K}$ is a signature and for any $\mathbf{A}\in\boldsymbol{K}$ and any set X, $\boldsymbol{K}$ contains the expansion $\mathbf{B}$ of $\mathbf{A}|(\Sigma\boldsymbol{K}-\{\mathrm{S}\})$ with $\mathbf{B}(\mathrm{S}) = X$. Suppose τ is the remainder of $\Sigma\boldsymbol{K}$ after all the sort declarations that are isolated with respect to $\boldsymbol{K}$ have been removed; we define $\boldsymbol{K}^\circ$ as $\tau\square\boldsymbol{K}$. Restricted to varieties, the assignment $\boldsymbol{K} \mapsto \boldsymbol{K}^\circ$ clearly preserves the ordering $\leq$; moreover $\boldsymbol{K} \leq \boldsymbol{K}^\circ$, and $\boldsymbol{K}^{\circ\circ}= \boldsymbol{K}$. It is easy to see that $\eta_{\boldsymbol{K}^\circ} = \eta_{\boldsymbol{K}}$.

7.12 Lemma. Let $\boldsymbol{K}$ and $\boldsymbol{L}$ be varieties. If there exists a morphism $\eta_{\boldsymbol{K}} \to \eta_{\boldsymbol{L}}$, then $\boldsymbol{K}^\circ \leq \boldsymbol{L}^\circ$.

Proof. Let $\sigma = \Sigma\boldsymbol{K}^\circ$, $\tau = \Sigma\boldsymbol{L}^\circ$, and $\alpha : \eta_{\boldsymbol{K}} \to \eta_{\boldsymbol{L}}$. By proposition 10, it suffices to show that $\mathrm{S}\sigma \subseteq \mathrm{S}\tau$. If $\mathrm{S} \in \mathrm{S}\sigma - \mathrm{S}\tau$, then since $\mathrm{F}\sigma \subseteq \mathrm{F}\tau$, S does not occur in function declarations in σ. Hence the only σ-terms that have subterms of sort S are variables. Hence all σ-equations about the sort S that are not universally valid, are logically equivalent, and have the form **x=y**, with **x** and **y** distinct. But if $\boldsymbol{K}\vDash\mathbf{x}=\mathbf{y}$, we must have $\boldsymbol{L}\vDash\mathbf{x}=\mathbf{y}$ as well; otherwise consider a two-element set X as an $\{\mathrm{S}\}$-algebra, then we cannot have a homomorphism

$$\alpha_X : F_{\boldsymbol{K}}(X) \to F_{\boldsymbol{L}}(X)$$

with $\alpha_X \circ \eta_{K,X} = \eta_{L,X}$. So S $\in$ Sσ after all. ¶

Let Var be the class of varieties, **Var** the semilattice $\langle Var, \leq \rangle$, and $H : \mathbf{Var} \to \mathbf{Un}(\mathbf{Alg})$ the assignment $\boldsymbol{K} \mapsto \eta_{\boldsymbol{K}}$. Observe that $\boldsymbol{K}^\circ + \boldsymbol{L}^\circ = (\boldsymbol{K} + \boldsymbol{L})^\circ$; we denote the subsemilattice of **Var** with universe $\{\boldsymbol{K}^\circ \mid \boldsymbol{K} \in \mathbf{Var}\}$ by $\mathbf{Var}^\circ$. We note that the definition of the uniformities $\eta_{\boldsymbol{M}}$ implies that isomorphic uniformities of this kind are actually identical. Hence the full subcategory $\mathbf{Sup} \leq \mathbf{Un}(\mathbf{Alg})$ determined by the superexpansions $\eta_{\boldsymbol{K}}$ is a semilattice. Now it is easy to prove, using lemmas 9 and 12:

7.13 Theorem. $\mathbf{Sup} \cong \mathbf{Var}^\circ$.

Let ϕ be a D-bijection; $\boldsymbol{K}$ a class of algebras and η a uniformity in **Alg**. We define $\phi \cdot \boldsymbol{K}$ as $\phi^*(\boldsymbol{K})$ and recall that $\phi \cdot \eta$ has been defined as $(\phi^{-1})^* \circ \eta \circ \phi^*$.

7.14 Lemma. Let $\boldsymbol{K}$ be a variety. Then $\eta_{\phi \cdot \boldsymbol{K}} \cong \phi^{-1} \cdot \eta_{\boldsymbol{K}}$.

Proof. Put $\eta = \eta_{\boldsymbol{K}}$, $F = F_{\boldsymbol{K}}$, $\theta = \eta_{\phi \cdot \boldsymbol{K}}$ and $G = F_{\phi \cdot \boldsymbol{K}}$. Let **A** be an algebra. Then $F(\mathbf{A} \circ \phi^{-1})$ is generated over $\boldsymbol{K}$, so $F(\mathbf{A} \circ \phi^{-1}) \circ \phi$ is generated over $\phi \cdot \boldsymbol{K}$. So $\phi^*(\eta_{\mathbf{A} \circ \phi^{-1}})$ factors through $\theta_{\mathbf{A}}$, say $\phi^*(\eta_{\mathbf{A} \circ \phi^{-1}}) = \alpha \circ \theta_{\mathbf{A}}$. On the other hand $G\mathbf{A}$ is generated over $\phi \cdot \boldsymbol{K}$, so $(\phi^{-1})^* G\mathbf{A} = G\mathbf{A} \circ \phi^{-1}$ is generated over $\boldsymbol{K}$; so $(\phi^{-1})^*(\theta_{\mathbf{A}})$ factors through $\eta_{\mathbf{A} \circ \phi^{-1}}$, say $(\phi^{-1})^*(\theta_{\mathbf{A}}) = \beta \circ \eta_{\mathbf{A} \circ \phi^{-1}}$.

$$
\begin{tikzcd}
\mathbf{A} \arrow[rr, "\theta"] \arrow[drr, "\phi^{-1} \cdot \eta"'] & & G\mathbf{A} \arrow[d, "\alpha"] \\
& & (\phi^{-1} \cdot F)(\mathbf{A})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{A} \circ \phi^{-1} \arrow[rr, "\eta"] \arrow[drr, "(\phi^{-1})^* \theta"'] & & F(\mathbf{A} \circ \phi^{-1}) \arrow[d, "\beta"] \\
& & G\mathbf{A} \circ \phi^{-1}
\end{tikzcd}
\tag{7.3}
$$

Observe that $\phi^*(\eta_{\mathbf{A} \circ \phi^{-1}}) = (\phi^{-1} \cdot \eta)_{\mathbf{A}}$. We have

$$\theta_{\mathbf{A}} = (\phi^* \circ (\phi^{-1})^*)(\theta_{\mathbf{A}}) = \phi^*(\beta \circ \eta_{\mathbf{A} \circ \phi^{-1}}) = \phi^*(\beta) \circ (\phi^{-1} \cdot \eta)_{\mathbf{A}}.$$

Consequently $\alpha \circ \phi^*(\beta) \circ (\phi^{-1} \cdot \eta)_{\mathbf{A}} = (\phi^{-1} \cdot \eta)_{\mathbf{A}}$; since ϕ^* is an automorphism of **Alg**, $(\phi^{-1} \cdot \eta)_{\mathbf{A}}$ is universal from $\mathbf{A} \circ \phi^{-1}$ to $\phi^* \circ F$; so $\alpha \circ \phi^*(\beta) = 1_{\phi^*(F(\mathbf{A} \circ \phi^{-1}))}$. Conversely, $\phi^*(\beta) \circ \alpha \circ \theta_{\mathbf{A}} = \phi^*(\beta) \circ \phi^*(\eta_{\mathbf{A} \circ \phi^{-1}}) = \phi^*(\beta \circ \eta_{\mathbf{A} \circ \phi^{-1}}) = \theta_{\mathbf{A}}$, so

$$\phi^*(\beta) \circ \alpha = 1_{G\mathbf{A}}$$

by the universal property of $\theta_{\mathbf{A}}$.

I will show now that the assignment $\mathbf{A} \mapsto \alpha$ (henceforth $\alpha_{\mathbf{A}}$) is natural. Take $f : \mathbf{A} \to \mathbf{B}$. Then we have:

$$\begin{tikzcd}
\mathbf{A} \arrow[rr, "f"] \arrow[dr, "\theta"] \arrow[ddr, "\phi^{-1}\cdot\eta"'] & & \mathbf{B} \arrow[dr, "\theta"] \arrow[ddr, dashed, "\phi^{-1}\cdot\eta"'] & \\
& G\mathbf{A} \arrow[rr, "Gf"] \arrow[d, "\alpha"] & & G\mathbf{B} \arrow[d, "\alpha"] \\
& (\phi^{-1}\cdot F)(\mathbf{A}) \arrow[rr, "(\phi^{-1}\cdot F)f"'] & & (\phi^{-1}\cdot F)(\mathbf{B})
\end{tikzcd} \tag{7.4}$$

$$\alpha_{\mathbf{B}}\circ Gf\circ\theta_{\mathbf{A}} = \alpha_{\mathbf{B}}\circ\theta_{\mathbf{B}}\circ f = (\phi^{-1}\cdot\eta)_{\mathbf{B}}\circ f = (\phi^{-1}\cdot F)f\circ(\phi^{-1}\cdot\eta)_{\mathbf{A}} = (\phi^{-1}\cdot F)f\circ\alpha_{\mathbf{A}}\circ\theta_{\mathbf{A}}$$

by naturality of θ, the definition of $\alpha_{\mathbf{B}}$ and naturality of $\phi^{-1}\cdot\eta$; hence by the universal property of $\theta_{\mathbf{A}}$, $\alpha_{\mathbf{B}}\circ Gf = (\phi^{-1}\cdot F)f\circ\alpha_{\mathbf{A}}$.

Likewise $\mathbf{A} \mapsto \beta_{\mathbf{A}}$ is natural from $F\circ(\phi^{-1})^*$ to $(\phi^{-1})^*\circ G$. We have:

$$\begin{tikzcd}
\mathbf{A}\circ\phi^{-1} \arrow[rr, "(\phi^{-1})^* f"] \arrow[dr, "\eta"] \arrow[ddr, "(\phi^{-1})^*\theta"'] & & \mathbf{B}\circ\phi^{-1} \arrow[dr, "\eta"] \arrow[ddr, dashed, "(\phi^{-1})^*\theta"'] & \\
& F(\mathbf{A}\circ\phi^{-1}) \arrow[rr, "F((\phi^{-1})^* f)"] \arrow[d, "\beta"] & & F(\mathbf{B}\circ\phi^{-1}) \arrow[d, "\beta"] \\
& G\mathbf{A}\circ\phi^{-1} \arrow[rr, "(\phi^{-1})^* Gf"'] & & G\mathbf{B}\circ\phi^{-1}
\end{tikzcd} \tag{7.5}$$

$$\begin{aligned}
\beta_{\mathbf{B}}\circ(F\circ(\phi^{-1})^*)f\circ\eta_{\mathbf{A}\circ\phi^{-1}} &= \beta_{\mathbf{B}}\circ\eta_{\mathbf{B}\circ\phi^{-1}}\circ(\phi^{-1})^* f \text{ by naturality of } \eta \\
&= (\phi^{-1})^*\theta_{\mathbf{B}}\circ(\phi^{-1})^* f \quad \text{by the definition of } \beta \\
&= (\phi^{-1})^*(\theta_{\mathbf{B}}\circ f) = (\phi^{-1})^*(Gf\circ\theta_{\mathbf{A}}) \text{ by naturality of } \theta \\
&= (\phi^{-1})^*(Gf)\circ(\phi^{-1})^*(\theta_{\mathbf{A}}) = (\phi^{-1})^*(Gf)\circ\beta_{\mathbf{A}}\circ\eta_{\mathbf{A}\circ\phi^{-1}} \text{ by the definition of } \beta,
\end{aligned}$$

so $\beta_{\mathbf{B}}\circ(F\circ(\phi^{-1})^*)f = (\phi^{-1})^*(Gf)\circ\beta_{\mathbf{A}}$ by the universal property of $\eta_{\mathbf{A}\circ\phi^{-1}}$. ¶

This proves in particular that the uniformities $\eta_{\boldsymbol{K}}$ are algebraic. Note that $\boldsymbol{K} = Mod\mathcal{A}$ implies $\phi\cdot\boldsymbol{K} = Mod(\phi^{-1}\cdot\mathcal{A})$.

Renamings are their own inverses, which makes things considerably easier:

7.15 Corollary. Let $\boldsymbol{K}$ be a variety and r a renaming. Then $\eta_{r\cdot\boldsymbol{K}} \cong r\cdot\eta_{\boldsymbol{K}}$.

8. Abstract uniformities

8.1 Definition. A *reduced uniformity* is a pair $\langle\sigma, \eta\rangle$ of a signature σ and an algebraic uniformity η, with $\sigma\subseteq\Sigma(\eta)$.

Let $\boldsymbol{\eta} = \langle\sigma, \eta\rangle$ and $\boldsymbol{\theta} = \langle\tau, \theta\rangle$, with $\eta : 1 \to F$, $\theta : 1 \to G$ and $\sigma\subseteq\tau$, be reduced uniformities. A *morphism* from $\boldsymbol{\eta}$ to $\boldsymbol{\theta}$ is a pair $\langle\phi, \alpha\rangle$ of a D-bijection ϕ with $\phi|\sigma = 1_\sigma$, and an **AUn**-morphism $\alpha : \eta \to \phi^{-1}\cdot\theta$. Then since $\Sigma(\phi^{-1}\cdot\theta) = \phi^{-1}\cdot\Sigma\theta$ (lemma 7.5) we must have $\phi\cdot\Sigma(\eta)\subseteq\Sigma(\theta)$. The composition of two such morphisms

$$\langle\sigma, \eta\rangle \xrightarrow{\langle\phi,\, \alpha\rangle} \langle\tau, \theta\rangle \xrightarrow{\langle\psi,\, \beta\rangle} \langle\upsilon, \zeta\rangle$$

is $\langle\psi\circ\phi,\ (\phi^{-1}\cdot\beta)\bullet\alpha\rangle$. This is indeed a morphism from $\langle\sigma, \eta\rangle$ to $\langle\upsilon, \zeta\rangle$: by 6.4 and 7.4(ii),

$$(\phi^{-1}\cdot\beta)\bullet\alpha\bullet\eta = (\phi^{-1}\cdot\beta)\bullet(\phi^{-1}\cdot\theta) = \phi^{-1}\cdot(\beta\bullet\theta) = \phi^{-1}\cdot(\psi^{-1}\cdot\zeta) = (\psi\phi)^{-1}\cdot\zeta\,.$$

Composition is associative: let

$$\langle\sigma, \eta\rangle \xrightarrow{\langle\phi,\, \alpha\rangle} \langle\tau, \theta\rangle \xrightarrow{\langle\psi,\, \beta\rangle} \langle\upsilon, \zeta\rangle \xrightarrow{\langle\chi,\, \gamma\rangle} \langle\rho, \varepsilon\rangle$$

be morphisms. Then

$$\langle\psi,\beta\rangle\circ\langle\phi, \alpha\rangle = \langle\psi\circ\phi,\ (\phi^{-1}\cdot\beta)\bullet\alpha\rangle \text{ and } \langle\chi, \gamma\rangle\circ\langle\psi, \beta\rangle = \langle\chi\circ\psi, (\psi^{-1}\cdot\gamma)\bullet\beta\rangle.$$

Composition of D-bijections is associative; so it suffices to note that by lemmas 7.4 and 6.4,

$$((\psi\phi)^{-1}\cdot\gamma)\bullet(\phi^{-1}\cdot\beta)\bullet\alpha = (\phi^{-1}\cdot((\psi^{-1}\cdot\gamma)\bullet\beta))\bullet\alpha\,.$$

The category of reduced uniformities with these morphisms (and identities $1_{\langle\sigma,\eta\rangle} = \langle 1, F\rangle$) will be denoted by **RUn**.

8.2 Definition. An *abstract uniformity* or *uniformity type* is an isomorphism class of objects of **RUn**.

I denote the isomorphism class of $\langle\sigma, \eta\rangle$ by (σ, η). Obviously, if $\langle\tau, \theta\rangle \cong \langle\sigma, \eta\rangle$, then $\tau=\sigma$.

8.3 Definition. Let $M := (\sigma, \eta)$ be an abstract uniformity; then the *signature of* M is defined by $\Sigma M = \sigma$.

8.4 Definition. Let $\mathcal{A} = \langle\sigma, \mathrm{A}\rangle$ be an equational theory; then $\langle\mathcal{A}\rangle = (\sigma, \eta_{\mathcal{A}})$.

It is clear that these definitions satisfy axiom scheme (S1):

(I) $\Sigma\langle\mathcal{A}\rangle = \Sigma\mathcal{A}$.

8.5 Definition. Let σ be a signature; then $\mathbb{T}(\sigma) = \langle\sigma, \eta_\sigma\rangle$, and T(σ) is the isomorphism class of $\mathbb{T}(\sigma)$.

In this way we satisfy axiom (S2):

(II) $\Sigma(\mathrm{T}(\sigma)) = \sigma$.

8.6 Definition. Let η and θ be algebraic uniformities, and σ a signature. We call η and θ σ-*equivalent*, notation $\eta \equiv_\sigma \theta$, if $\langle\sigma, \eta\rangle \cong \langle\sigma, \theta\rangle$ in **RUn**.

The construction of σ-equivalent uniformities is not difficult:

8.7 Lemma. Let $\eta : 1 \to F$ be an algebraic uniformity, $\sigma \subseteq \Sigma(\eta)$ and r a renaming with the property that $r|\sigma = 1_\sigma$. Then $\langle r, F\rangle$ is an isomorphism from $\langle\sigma, \eta\rangle$ to $\langle\sigma, r\cdot\eta\rangle$.

Proof. We recall that r is an involution, and calculate:

$$\langle r, F\rangle \circ \langle r, r\cdot F\rangle = \langle r\circ r, (r\cdot F)\bullet(r\cdot F)\rangle = \langle 1, r\cdot F\rangle = 1_{\langle\sigma, r\cdot\eta\rangle}, \text{ and}$$
$$\langle r, r\cdot F\rangle \circ \langle r, F\rangle = \langle r\circ r, r\cdot(r\cdot F)\bullet F\rangle = \langle 1, F\bullet F\rangle = \langle 1, F\rangle = 1_{\langle\sigma,\eta\rangle}. \quad ¶$$

8.8 Lemma. Let $\langle\sigma, \eta\rangle$ and $\langle\tau, \theta\rangle$ be reduced uniformities. Then there exist algebraic uniformities η' and θ' such that $\eta \equiv_\sigma \eta'$, $\theta \equiv_\tau \theta'$, and $\Sigma(\eta')\cap\Sigma(\theta') \subseteq \sigma\cap\tau$.

Proof. Apply lemma 2.5 with $\upsilon = \Sigma(\eta)$ en $\phi = \Sigma(\theta)$, to obtain renamings r and s such that $r|\sigma = 1_\sigma$, $s|\tau = 1_\tau$, and $r\cdot\Sigma(\eta)\cap s\cdot\Sigma(\theta) \subseteq \sigma\cap\tau$. Take $\eta' = r\cdot\eta$ and $\theta' = s\cdot\theta$, and use the preceding lemma. ¶

8.9 Lemma. Let $\langle\sigma,\eta\rangle$ and $\langle\tau,\theta\rangle$ be reduced uniformities, with $\sigma\subseteq\tau$, and $\alpha : \eta \to \theta$. Then $\langle 1, \alpha\rangle$ is a morphism from $\langle\sigma,\eta\rangle$ to $\langle\tau,\theta\rangle$.

Proof. Trivial. ¶

8.10 Definition. Let $\langle\sigma,\eta\rangle$ and $\langle\tau,\theta\rangle$ be reduced uniformities, such that $\Sigma(\eta)\cap\Sigma(\theta) = \sigma\cap\tau$. Then $\langle\sigma,\eta\rangle + \langle\tau,\theta\rangle = \langle\sigma\cup\tau,\ \eta+\theta\rangle$.

8.11 Lemma. (i) If $\langle\phi, \alpha\rangle : \langle\sigma, \eta\rangle \to \langle\sigma', \eta'\rangle$, and α is invertible, then so is $\langle\phi, \alpha\rangle$, and $\langle\phi,\alpha\rangle^{-1} = \langle\phi^{-1}, \phi\cdot\alpha^{-1}\rangle$.
(ii) If $\langle\phi, \alpha\rangle : \langle\sigma, \eta\rangle \cong \langle\sigma', \eta'\rangle$, then $\langle\phi, \alpha\rangle^{-1} = \langle\phi^{-1}, \phi\cdot\alpha^{-1}\rangle$.

Proof. (i) Let F and F' be the respective codomains of η and η'; then α is a natural isomorphism from F to $\phi^{-1}{\cdot}F'$. We have:

$$\begin{aligned}\langle\phi,\alpha\rangle\circ\langle\phi^{-1},\phi\cdot\alpha^{-1}\rangle &= \langle 1,(\phi\cdot\alpha)\bullet(\phi\cdot\alpha^{-1})\rangle = \langle 1,\phi\cdot(\alpha\bullet\alpha^{-1})\rangle \\ &= \langle 1,\phi\cdot(\phi^{-1}\cdot F')\rangle = \langle 1,F'\rangle = 1_{\langle\sigma',\eta'\rangle},\end{aligned}$$

and

$$\begin{aligned}\langle\phi^{-1},\phi\cdot\alpha^{-1}\rangle\circ\langle\phi,\alpha\rangle &= \langle 1,(\phi^{-1}\cdot(\phi\cdot\alpha^{-1}))\bullet\alpha\rangle = \langle 1,\alpha^{-1}\bullet\alpha\rangle = \langle 1,F\rangle \\ &= 1_{\langle\sigma,\eta\rangle}.\end{aligned}$$

(ii) Suppose $\langle\phi,\alpha\rangle^{-1} = \langle\psi,\beta\rangle$. Then $\psi\phi = 1$, so $\psi = \phi^{-1}$. Moreover

$$\langle\phi,\alpha\rangle\circ\langle\psi,\beta\rangle = \langle 1,F'\rangle,$$

so $(\phi\cdot\alpha)\bullet\beta = F'$, i.e. $\alpha\bullet(\phi^{-1}\cdot\beta) = \phi^{-1}\cdot F'$. Also $\langle\psi,\beta\rangle\circ\langle\phi,\alpha\rangle = \langle 1,F\rangle$, so $(\phi^{-1}\cdot\beta)\bullet\alpha = F$. So $\phi^{-1}\cdot\beta = \alpha^{-1}$, and $\beta = \phi\cdot\alpha^{-1}$. ¶

8.12 Lemma. If $\langle\phi,\alpha\rangle:\langle\sigma,\eta\rangle\cong\langle\sigma',\eta'\rangle$, and $\chi|\Sigma(\eta) = \phi|\Sigma(\eta)$, then there exists an isomorphism $\langle\chi,\alpha'\rangle:\langle\sigma,\eta\rangle\cong\langle\sigma',\eta'\rangle$.

Proof. According to lemma 11, $\langle\phi,\alpha\rangle$ has inverse $\langle\phi^{-1},\phi\cdot\alpha^{-1}\rangle$.

$$\begin{tikzcd} & 1 \arrow[dl, "\eta"'] \arrow[dr, "\phi^{-1}\eta'"] & \\ F \arrow[rr, "\alpha"'] & & \phi^{-1}F' \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dl, "\phi\eta"'] \arrow[dr, "\eta'"] & \\ \phi F & & F' \arrow[ll, "\phi\alpha^{-1}"] \end{tikzcd} \tag{8.1}$$

Since η is algebraic, and $\chi|\Sigma(\eta) = \phi|\Sigma(\eta)$, there exists an isomorphism δ from $\phi\cdot\eta$ to $\chi\cdot\eta$. Set $\omega=\chi^{-1}\phi$, and consider the (commuting) diagrams 8.2.

$$\begin{tikzcd} & & 1 \arrow[dll, "\chi\eta"'] \arrow[d, "\phi\eta"] \arrow[drr, "\eta'"] & & \\ \chi F \arrow[rr, "\delta^{-1}"'] & & \phi F \arrow[rr, shift right, "\phi\alpha"'] & & F' \arrow[ll, shift right, "\phi\alpha^{-1}"'] \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dl, "\eta"'] \arrow[d, "\omega\eta"] \arrow[drr, "\chi^{-1}\eta'"] & & \\ F \arrow[r, "\chi^{-1}\delta^{-1}"'] & \omega F \arrow[rr, "\omega\alpha"'] & & \chi^{-1}F' \end{tikzcd} \tag{8.2}$$

Put $\alpha'=\omega\alpha\bullet\chi^{-1}\delta^{-1}$. By the right-hand diagram

$$\langle\chi,\alpha'\rangle:\langle\sigma,\eta\rangle\to\langle\sigma,\eta'\rangle.$$

The transformation α' is invertible:

$$\begin{aligned}\omega\alpha\bullet\chi^{-1}\delta^{-1}\bullet\chi^{-1}\delta\bullet\omega\alpha^{-1} &= \omega\alpha\bullet\chi^{-1}\cdot(\delta^{-1}\bullet\delta)\bullet\omega\alpha^{-1} = \omega\alpha\bullet\omega F\bullet\omega\alpha^{-1} \\ &= \omega\alpha\bullet\omega\alpha^{-1} = \omega\cdot(\alpha\bullet\alpha^{-1}) = \omega\cdot(\phi^{-1}F) = (\omega\circ\phi^{-1})\cdot F \\ &= \chi^{-1}F\end{aligned}$$

and $\chi^{-1}\delta\bullet\omega\alpha^{-1}\bullet\omega\alpha\bullet\chi^{-1}\delta^{-1} = \chi^{-1}\delta\bullet\omega F\bullet\chi^{-1}\delta^{-1} = \chi^{-1}\delta\bullet\chi^{-1}\delta^{-1} = \chi^{-1}\cdot(\chi F) = F$. So $\langle\chi,\alpha'\rangle$ has inverse $\langle\chi^{-1},\chi\cdot(\alpha')^{-1}\rangle$, by lemma 11. ¶

8.13 Corollary. If $\eta\equiv_\sigma\eta'$ and $\theta\equiv_\tau\theta'$, and

$$(\Sigma(\eta)\cap\Sigma(\theta)) \cup (\Sigma(\eta')\cap\Sigma(\theta')) \subseteq \sigma\cap\tau,$$

then $\langle\sigma\cup\tau,\ \eta+\theta\rangle \cong \langle\sigma\cup\tau,\ \eta'+\theta'\rangle$.

Proof. Suppose $\langle\phi,\alpha\rangle : \langle\sigma,\eta\rangle\cong\langle\sigma,\eta'\rangle$ and $\langle\psi,\beta\rangle : \langle\tau,\theta\rangle\cong\langle\tau,\theta'\rangle$. Then

$$\chi_0 := \phi|\Sigma(\eta)\cup\psi|\Sigma(\theta)$$

is an injective signature morphism. Let χ be a D-bijection extending χ_0 (χ exists by lemma 2.2). Then $\chi|(\sigma\cup\tau) = \phi|\sigma\cup\psi|\tau = 1_{\sigma\cup\tau}$. By lemma 12 we have isomorphisms $\langle\chi,\alpha'\rangle : \langle\sigma,\eta\rangle\cong\langle\sigma,\eta'\rangle$ and $\langle\chi,\beta'\rangle : \langle\tau,\theta\rangle\cong\langle\tau,\theta'\rangle$. Using the observation preceding lemma 7.6, we find an isomorphism γ from $\eta+\theta$ to $\chi^{-1}\cdot(\eta'+\theta')$, and then by lemma 11,

$$\langle\chi,\gamma\rangle : \langle\sigma\cup\tau,\ \eta+\theta\rangle\cong\langle\sigma\cup\tau,\ \eta'+\theta'\rangle. \qquad ¶$$

8.14 Definition. Let $M:=(\sigma,\eta)$ and $N:=(\tau,\theta)$ be uniformity types; suppose $\Sigma(\eta)\cap\Sigma(\theta) = \sigma\cap\tau$. Then $M+N = (\sigma\cup\tau,\ \eta+\theta)$.

The foregoing implies that this unambiguously determines the sum of any pair of uniformity types: by lemma 8, if M or N happen to be given by means of representatives that do not satisfy the condition that $\Sigma(\eta)\cap\Sigma(\theta)\subseteq\sigma\cap\tau$, better representatives may be found; and by corollary 13 the choice of representatives, as long as they satisfy the condition, does not matter. The definition implies immediately:

(III) $\Sigma(M+N) = \Sigma M\cup\Sigma N$.

8.15 Definition. If $\langle\sigma,\eta\rangle$ is a reduced uniformity and τ a signature, then $\tau\Box\langle\sigma,\eta\rangle = \langle\sigma\cap\tau,\eta\rangle$, and $\tau\Box(\sigma,\eta) = (\sigma\cap\tau,\eta)$.

It is easy to see that this definition is correct; in particular an isomorphism from η to θ is also an isomorphism from $\sigma\Box\eta$ to $\sigma\Box\theta$. Observe that

(IV) $\Sigma(\sigma\Box M) = \sigma\cap\Sigma M$.

8.16 Lemma. Let ϕ be a D-bijection and $\langle\psi,\alpha\rangle : \langle\sigma,\eta\rangle\to\langle\tau,\theta\rangle$. Then

$$\langle\phi\circ\psi\circ\phi^{-1},\phi\cdot\alpha\rangle : \langle\phi\cdot\sigma,\ \phi\cdot\eta\rangle\to\langle\phi\cdot\tau,\ \phi\cdot\theta\rangle.$$

Proof. For $\mathbf{d} \in \sigma$, $(\phi \circ \psi \circ \phi^{-1})(\phi \cdot \mathbf{d}) = \phi(\psi(\mathbf{d})) = \phi \cdot \mathbf{d}$, since $\psi | \sigma = 1_\sigma$. For the rest, using lemma 7.4 we calculate:

$$\begin{aligned}(\phi\cdot\alpha)\bullet((\phi\circ\psi^{-1}\circ\phi^{-1})\cdot(\phi\cdot\eta) &= (\phi\cdot\alpha)\bullet((\phi\circ\psi^{-1}\circ\phi^{-1}\circ\phi)\cdot\eta)\\ &= (\phi\cdot\alpha)\bullet((\phi\circ\psi^{-1})\cdot\eta)\\ &= \phi\cdot(\alpha\bullet(\psi^{-1}\cdot\eta)) \qquad (6.4)\\ &= \phi\theta. \qquad ¶\end{aligned}$$

We define: $\phi\cdot\langle\psi, \alpha\rangle = \langle\phi\circ\psi\circ\phi^{-1}, \phi\cdot\alpha\rangle$.

8.17 Lemma. The assignment $\langle\psi, \alpha\rangle \mapsto \phi\cdot\langle\psi, \alpha\rangle$ is an endofunctor of **RUn**.

Proof. In particular:

$$\begin{aligned}\phi\cdot(\langle\chi, \beta\rangle\circ\langle\psi, \alpha\rangle) = \phi\cdot\langle(\chi\psi, (\psi^{-1}\beta)\bullet\alpha)\rangle &= \langle\phi\circ\chi\psi\circ\phi^{-1}, \phi\cdot((\psi^{-1}\beta)\bullet\alpha))\rangle\\ &= \langle\phi\chi\phi^{-1}\circ\phi\psi\phi^{-1}, (\phi\cdot(\psi^{-1}\beta))\bullet(\phi\cdot\alpha)\rangle\\ &= \langle\phi\chi\phi^{-1}, \phi\beta\rangle\circ\langle\phi\psi\phi^{-1}, \phi\alpha\rangle\\ &= (\phi\cdot\langle\chi, \beta\rangle)\circ(\phi\cdot\langle\psi, \alpha\rangle),\end{aligned}$$

because $\phi\cdot(\psi^{-1}\beta) = (\phi\circ\psi^{-1}\circ\phi^{-1})\cdot(\phi\beta) = (\phi\psi\phi^{-1})^{-1}\cdot\phi\beta$. ¶

Hence we may define $\phi\cdot(\sigma, \eta)$ as $(\phi\cdot\sigma, \phi\cdot\eta)$. We conclude immediately, for renamings r:

(V) $\Sigma(r\cdot M) = r\cdot\Sigma M$

(VI) $r\cdot\langle\mathcal{A}\rangle = \langle r\cdot\mathcal{A}\rangle$

(VII) $r\cdot\mathbb{T}(\sigma) = \mathbb{T}(r\cdot\sigma)$

Let r be a renaming. As was noted above 7.6, $r\cdot(\eta+\theta)$ is isomorphic to $r\cdot\eta+r\cdot\theta$; by lemma 9 this implies

$$\langle r\cdot(\sigma\cup\tau), r\cdot(\eta+\theta)\rangle \cong \langle r\cdot\sigma\cup r\cdot\tau, r\cdot\eta+r\cdot\theta\rangle,$$

hence

(VIII) $r\cdot(M+N) = r\cdot M+r\cdot N$.

We have

$$\begin{aligned}r\cdot(\sigma\square\langle\tau, \theta\rangle) = r\cdot\langle\sigma\cap\tau, \theta\rangle) &= \langle r\cdot(\sigma\cap\tau), r\cdot\theta\rangle) = \langle r\cdot\sigma\cap r\cdot\tau, r\cdot\theta\rangle)\\ &= (r\cdot\sigma)\square(r\cdot\langle\tau, \theta\rangle).\end{aligned}$$

Hence

(IX) $r\cdot(\sigma\square N) = (r\cdot\sigma)\square(r\cdot N)$.

For $\eta \in RUn$, $r{\cdot}r{\cdot}\eta = \eta$; so

(X) $\quad r{\cdot}(r{\cdot}M) = M$.

We have $r|(\sigma/r) = 1_{\sigma/r}$ by definition. Hence $r{\cdot}((\sigma/r)\square\langle\tau,\theta\rangle) = \langle(\sigma/r)\cap\tau, r{\cdot}\theta\rangle$; and if G is the codomain of θ, $\langle r, r{\cdot}G\rangle$ is an isomorphism from $\langle(\sigma/r)\cap\tau, r{\cdot}\theta\rangle$ to $\langle(\sigma/r)\cap\tau, \theta\rangle$. Hence

(XI) $\quad r{\cdot}((\sigma/r)\square N) = (\sigma/r)\square N$.

From lemma 9 and commutativity and associativity modulo isomorphism of the coproduct operation we see

(XII) $\quad M+N = N+M$;
(XIII) $\quad (L+M)+N) = L+(M+N)$.

A special case of theorem 7.11 is: $\eta_{\sigma\cup\tau} \cong \eta_\sigma + \eta_\tau$. Hence by lemma 9:

(XIV) $\quad \mathrm{T}(\sigma\cup\tau) = \mathrm{T}(\sigma)+\mathrm{T}(\tau)$.

Let us call the codomain of an arrow with domain a, in a given category $\mathbf{C}$, a *co-object* of a (in $\mathbf{C}$).

8.18 Lemma. If $\sigma\subseteq\Sigma\eta$, then $\eta+\eta_\sigma \cong \eta$.

Proof. Let $\mathbf{A}$ be an algebra. Then for each element t of $\mathbf{T}{:=}T_\sigma(\mathbf{A})$ there exists a unique sequence $\langle \mathbf{t}, a_1,\ldots,a_n\rangle$ of a term $\mathbf{t}$ over $\sigma-\Sigma\mathbf{A}$ and elements $a_1,\ldots,a_n$ of A such that $t = \mathbf{T}(\mathbf{t})(a_1,\ldots,a_n)$. It is evident that every homomorphism $\phi{:}\,\mathbf{A}\to\mathbf{B}$ to a σ-generated algebra $\mathbf{B}$ has exactly one extension over $\mathbf{T}$. In particular, let F be the codomain of η; define $\rho_{\mathbf{A}}:\mathbf{T}\to F\mathbf{A}$ by

$$\rho_{\mathbf{A}}(\mathbf{T}(\mathbf{t})(a_1,\ldots,a_n)) = F\mathbf{A}(\mathbf{t})(\eta_{\mathbf{A}}(a_1),\ldots,\eta_{\mathbf{A}}(a_n)),$$

Then ρ is an $\mathbf{AUn}$-morphism from η_σ to η. We shall be done once we see that η is a coproduct of η and η_σ, with injections 1_η $(= F)$ en ρ.

Let $\theta{:}\,1\to G$ be given, with morphisms $\alpha{:}\,\eta\to\theta$ and $\beta{:}\,\eta_\sigma\to\theta$.

$$\begin{tikzcd}
\eta \arrow[r, "1"] \arrow[dr, "\alpha"'] & \eta \arrow[d] & \eta_\sigma \arrow[l, "\rho"'] \arrow[dl, "\beta"] \\
& \theta &
\end{tikzcd}$$

Proving that the pair $\langle\alpha,\beta\rangle$ factors uniquely through $\langle F,\rho\rangle$ is tantamount to showing that $\beta = \alpha\bullet\rho$. But this follows immediately by the unicity, for each algebra **A**, of homomorphisms from $T_\sigma(\mathbf{A})$ to co-objects of **A**. ¶

By lemma 9 we now have $\langle\sigma,\eta\rangle + \langle\sigma,\eta_\sigma\rangle \cong \langle\sigma,\eta\rangle$, hence

(XV) $M + \mathrm{T}\Sigma M = M$.

As an immediate consequence of theorem 7.11 we get

(XVI) $\langle\mathcal{A}\rangle + \langle\mathcal{B}\rangle = \langle\mathcal{A}+\mathcal{B}\rangle$.

The identities

(XVII) $\Sigma M \,\square\, M = M$,

(XVIII) $\sigma\,\square\,(\tau\,\square\,M) = (\sigma\cap\tau)\,\square\,M$

are immediate consequences of the definitions. Finally,

$$\begin{aligned}\sigma\,\square\,\langle\sigma\cup\tau,\,\eta+\theta\rangle &= \langle\sigma,\,\eta+\theta\rangle = \langle\sigma\cup(\sigma\cap\tau),\eta+\theta\rangle\\ &= \langle\sigma,\,\eta\rangle + \sigma\,\square\,\langle\tau,\,\theta\rangle;\end{aligned}$$

i.e. in terms of uniformity types,

(XIX) $\Sigma M\,\square\,(M+N) = M + \Sigma M\,\square\,M$.

Taken together, the formulas (I-XIX) imply:

8.19 Theorem. Signatures and uniformity types constitute a module algebra.

Let $\varnothing$ be the void algebra, more precisely: the algebra of void signature. If $\Sigma\mathcal{A}$ is inhabited, then $F_{\mathcal{A}}(\varnothing)$ is an initial object in the full subcategory of **Alg** determined by the object class $Mod(\mathcal{A})$. The fact that $\varnothing$ is initial in **Alg** even implies, through the universal property of $\eta_{\mathcal{A},\varnothing}$, that $F_{\mathcal{A}}(\varnothing)$ is initial in the class $\{F_{\mathcal{A}}(\mathbf{B}) \mid \mathbf{B}\in\mathbf{Alg}\}$, which contains $Mod(\mathcal{A})$ since for $\mathbf{B}\in Mod(\mathcal{A})$, $F_{\mathcal{A}}(\mathbf{B}) = \mathbf{B}$.

9 Counterexamples

The most typical of the laws of $BMA[fol]$ that are not valid for uniformities is the idempotency law $X+X=X$. It is immediately odd from the point of view of initial algebra semantics: if we specify twice a hidden object of sort S, then surely we end up with two objects. And indeed, it is not hard to see that

$$\{\mathrm{S}\}\;\square\;\langle\{\mathrm{S},\,\langle\mathrm{x},\mathrm{S}\rangle\},\,\varnothing\rangle$$

is a counterexample to idempotency. A fortiori $X + y\,\square\,X = X$, (C5) in [BHK], is not valid. Likewise, if y introduces a constant, of a sort in X, that cannot be reduced to something given by X, $\Sigma X \square (\mathrm{T}y+X) \neq X$; and similarly we may get $x \square \mathrm{T}y \neq \mathrm{T}(x \cap y)$. Either one of these inequalities contradicts the *BMA*[*fol*]-axiom (E3),

$$x \square (\mathrm{T}y+Z) = \mathrm{T}(x \cap y) + x \square Z.$$

There may be interesting further characterizations of the class of uniformities that are denoted by closed module expressions of $BMA_0[eql]$. It is easy enough to find a uniformity that is not thus denoted: consider adding a constant c of sort S to the input algebra **A** on condition that $\mathrm{S} \in \Sigma\mathbf{A}$; but this is also too easily excluded.

10 Complexity and completeness

Mechanical procedures for deciding whether two arbitrary specifications have the same meaning have not been among our most prominent objectives. Nevertheless, we should like to have some idea of what we are up against.

Assume equality on the set of names N is decidable, and consider *finite* equational theories only. Let us ask ourselves what is the recursion-theoretic complexity of the decision problem for closed module equations, when module terms are interpreted in accordance with §8.

There is no gauging the complexity of operations (such as modules are, on our interpretation) without assumptions on the input. I will assume in any case that the input algebras are *countable*; we recall that their signatures are finite. In what way we may consider such algebras to be given, is explained in some detail in [BT]; in effect, arbitrary countable algebras of finite signature may be represented by arbitrary functions from $\mathbb{N}$ to $\mathbb{N}$.

By the normal form theorem (proposition 4.3) and its corollary we may assume that module expressions have the form $\sigma \square \langle \mathcal{A} \rangle$, for a signature σ and a finite equational theory $\mathcal{A}$. The status of the hidden signature $\Sigma\mathcal{A} - \sigma$ is simply this: when an input algebra **C** is given, we must supply fresh names for the items in the overlap of $\Sigma\mathcal{A} - \sigma$ and $\Sigma\mathbf{C}$. Likewise, should there be hidden signature in a specification of **C** that we happen to use, we must make sure that this signature does not become an unintended interface. It is easy enough to imagine uniform, effective renaming procedures that take care of this; reason to abstract from the problems of hidden signature.

Now the target algebra $F_{\mathcal{A}}(\mathbf{C})$ may be represented by the equations

$$(*) \qquad \mathbf{s}(\mathbf{c}_1,\ldots,\mathbf{c}_n) = \mathbf{t}(\mathbf{c}_1,\ldots,\mathbf{c}_n)$$

over $\Sigma\mathcal{A}$ in parameters $\mathbf{c}_1,\ldots,\mathbf{c}_n$ from **C** that have a derivation in $\mathcal{A} + \mathcal{E}q\mathbf{C}_C$. The statement that (*) is derivable has the form $P(f, m)$, where P is a recursively enumerable relation (cf. [S]) representing derivability in $\mathcal{A}$, the algebra $\mathbf{C}$ is coded in the function argument f, and m is a number coding the terms $\mathbf{s}$ and $\mathbf{t}$ and — through f — the parameters $\mathbf{c}_1,\ldots,\mathbf{c}_n$. If $\langle\mathcal{B}\rangle$ is another module, with derivability relation Q, then the statement that $\langle\mathcal{A}\rangle$ and $\langle\mathcal{B}\rangle$ are equal may be made in the form

(1) $\forall f \forall m(P(f,m) \leftrightarrow Q(f,m))$.

This shows that Π^1_1 is an upper bound for the complexity of the problem. It may be simpler, for all I know.

If we assume that the input algebras are given by initial algebra specifications, we get a (much lower) precise estimate. In this case the function parameter may be replaced by a numerical code e for a partial recursive function $\{e\}$. Then $\langle\mathcal{A}\rangle$ and $\langle\mathcal{B}\rangle$ are equal if

(1) $\forall e \forall m(P(\{e\},m) \leftrightarrow Q(\{e\},m))$.

Rewriting $P(\{e\},m)$ in the form $\exists x P_0(x,e,m)$, for recursive P_0, and likewise for Q, we get

(2) $\forall e \forall m(\exists x P_0(x,e,m) \leftrightarrow \exists x Q_0(x,e,m))$,

which is at most Π^0_2. Now the bottom line is where the input algebra is $\varnothing$; then the problem is simply to decide of two numbers e_1 and e_2 whether the partial functions $\{e_1\}$ and $\{e_2\}$ have the same domain of definition. But if some oracle decides this problem for us, then we can decide for any e whether the domain of $\{e\}$ is infinite: for suppose E is the procedure coded by e; we can effectively construct a code $f(e)$ for the following procedure F on input n: apply E alternately for an increasing number of steps to 0, 1, 0, 1, 2, 0, 1, 2, 3, 0,…; output the $(n+1)$-th input on which you observe E to terminate. If we can decide whether $\{f(e)\}=\{a\}$, for some given code a of $1_{\mathbb{N}}$, then we know at once that $\{e\}$ has infinite domain. But the set of e such that $\{e\}$ has infinite domain is Π^0_2-complete (e.g. by [S], exercise 9 of chapter 7); so the matter we began with is also *at least* Π^0_2.

Another kind of question, related to the questions of complexity by its recursion theoretic nature, concerns the completeness of algebraic specification methods with respect to algebraic uniformities: is it the case that all effective (in senses to be specified) algebraic uniformities have equational (or conditional equational, or whatever else may come up) specifications? These questions are obviously related to completeness questions for parametrized specifications as treated in [BK], [R] and [Mo]. In our case there is no prescribed parameter signature and no requirement of persistency.

11 Parametrization

The concepts of specification theory — import, export, parameter passing etc. — have their meanings first and foremost in the practice of specification, apart from any mathematical interpretation; and in the second instance, from the mathematical interpretation of simple cases: almost-flat modules, simple and preferably persistent imports. We should not be surprised when such concepts turn out to have more than one continuation to nontrivial cases. Perhaps export is the clearest example at hand of this phenomenon: on one interpretation presented in [BHK], by model classes, hiding is existential quantification; on another (theories) it is reduction.

One could even say that in [BHK] the concepts of import and parametrization have evaporated. They do not occur as such, we have only more abstract operations from which they may be built up. The construction of import is fairly obvious; but parameter passing, which the authors modestly say they leave for another day, may easily be thought of as a combination of import (of a specification of a parameter) and renaming. On top of this, the present interpretation allows an especially straightforward form of parameter passing as application of the designated functor — or indirectly, as composition of functors. In any case, it has been shown that, though there may be many conditions under which parameter passing has various desirable properties, the operation by itself may be defined without reference to any of them.

Acknowledgments

Jan Bergstra encouraged me to work out the ideas that led to the investigation here reported. Bas van Vlijmen helped to overthrow some early illusions.

Appendix: categories, natural transformations and colimits

Categories may be regarded as two-sorted partial algebras. A category $\mathbf{C}$ has a sort $\mathbf{C}(0)$ (or $\mathbf{C}_0$) of *objects* and a sort $\mathbf{C}(1)$ (or $\mathbf{C}_1$) of *arrows* (also called *morphisms*); total operations id: $\mathbf{C}_0 \to \mathbf{C}_1$ (*identity*; id(x) will be written 1_x almost always) and dom, cod: $\mathbf{C}_1 \to \mathbf{C}_0$ (*domain* and *codomain*); and a partial operation of *composition* $\mathbf{C}_1 \circ \mathbf{C}_1 \to \mathbf{C}_1$, with the property that $f \circ g$ exists if and only if $\mathrm{dom} f = \mathrm{cod} g$. Composition is associative, $\mathrm{dom} 1_x = \mathrm{cod} 1_x = x$,

$\mathrm{dom}(f \circ g) = \mathrm{dom}\, g$ and $\mathrm{cod}(f \circ g) = \mathrm{cod}\, f$[1], and $1_{\mathrm{cod}\, f} \circ f = f \circ 1_{\mathrm{dom}\, f} = f$. The composition sign is often dropped. That f is an arrow with domain a and codomain b is expressed succinctly by $f : a \to b$.

The objects of a category are not strictly necessary: since they are in 1-1 correspondence with the identity arrows, it does not make any real difference if we identify them with their identity arrows. They have some use in that they provide an efficient notation for composability.

Occasionally it is useful to speak of a group or an ordering whose elements form a proper class; with categories such occasions abound. A prime example of a 'large' category is the category **Set**, whose objects are all sets (this could be specified further), and whose arrows are all triples $\langle X, f, Y \rangle$ in which X and Y are sets and f is a function from X to Y (i.e. a single-valued subrelation of $X \times Y$ such that $D(f) = X$ and $C(f) \subseteq Y$). It is common practice to mention only f in referring to the triple $\langle X, f, Y \rangle$, but the codomain does make a difference for composability. The identity 1_X is

$$\langle X,\ \{\langle x, x \rangle \mid x \in X\},\ X \rangle;$$

$\mathrm{dom}(X, f, Y) = X$, $\mathrm{cod}(X, f, Y) = Y$, and the composite $g \circ f$ of $\langle Y, g, Z \rangle$ and $\langle X, f, Y \rangle$ is

$$\langle X,\ \{\langle x, z \rangle \mid \exists y \in Y(\langle x, y \rangle \in f \ \&\ \langle y, z \rangle \in g)\},\ Z \rangle.$$

Another large example is the category **Alg** defined in §5.

A1. Further examples

1° Let $\mathbf{M} = \langle M, 1, \cdot \rangle$ be a monoid. Then $\mathbf{M}$ may be regarded as a category with a single object, with identity 1, and arrow sort M; the composition of $m, n \in M$ is $m \cdot n$.

2° Let $\mathbf{L} = \langle L, \prec \rangle$ be a quasi-ordering — in other words, let $\prec$ be a reflexive and transitive relation on the set L. Then $\mathbf{L}$ determines a category with object sort L and arrow sort $\prec$ ($= \{\langle x, y \rangle \in L \times L \mid x \prec y\}$), in which x is the domain of $\langle x, y \rangle$ and y the codomain, x has identity $\langle x, x \rangle$, and $\langle y, z \rangle \circ \langle x, y \rangle = \langle x, z \rangle$. By an *ordering* I shall understand an antisymmetric quasi-ordering, that is, what is commonly called a *partial* ordering.

3° The domain and codomain operations mirror each other. If in a given category $\mathbf{C}$ we interchange them, inverting at the same time the order of composability, we get another category, the *opposite* category $\mathbf{C}^{\mathrm{op}}$. Any arrow $f : a \to b$ of $\mathbf{C}$

[1] On condition that $f \circ g$ exists, of course; in practice one writes down a composite only if it exists.

becomes an arrow $b \to a$ in $\mathbf{C}^{\mathrm{op}}$. Opposite construction leads to categorical duality: every concept has a dual, often indicated by the præfix co-. A co-whatsit in $\mathbf{C}$ is a whatsit in $\mathbf{C}^{\mathrm{op}}$; domains are a case in point.

A *left inverse* of an arrow $f: a \to b$ is an arrow $g : b \to a$ such that $gf = 1_a$. Dually, f is a *right inverse* of g. An *inverse* of an arrow h is an arrow that is both left and right inverse to h; an arrow that has an inverse is an *isomorphism*, and its domain and codomain are *isomorphic*. (Note that this terminology is sound with respect to sets and algebras.) With reference to example 1.1° above, a one-object category in which every arrow has an inverse is a group.

A *monomorphism* is an arrow f such that for any arrows g, h, $fg = fh$ implies that $g = h$. The dual notion is that of an *epimorphism*. In **Set**, the monomorphisms are the injections, and the epimorphisms the surjections. In **Alg**, a homomorphism is a monomorphism if it is injective, and an epimorphism if it generates its codomain.

Within reasonable limits[1], the categories may be considered as a particular category **Cat** of partial algebras. The arrows of this category are called *functors*. A functor $F : \mathbf{C} \to \mathbf{D}$ maps every object c of $\mathbf{C}$ to an object Fc of $\mathbf{D}$, and every arrow $f: a \to b$ to an arrow $Ff: Fa \to Fb$ in such a way that $F(1_c) = 1_{Fc}$ and $F(f \circ g) = Ff \circ Fg$. The category $\mathbf{C}$ is a *subcategory* of $\mathbf{D}$ (notation, naturally, $\mathbf{C} \le \mathbf{D}$) if $\mathbf{C}_0 \subseteq \mathbf{D}_0$, $\mathbf{C}_1 \subseteq \mathbf{D}_1$, and the operations of $\mathbf{C}$ are the restrictions of the operations of $\mathbf{D}$. We call $\mathbf{C}$ a *full* subcategory if every arrow of $\mathbf{D}$ whose domain and codomain belong to $\mathbf{C}$ belongs to $\mathbf{C}$. Note that a full subcategory of a given category $\mathbf{D}$ is completely determined by its object sort.

Let F and G be functors from $\mathbf{C}$ to $\mathbf{D}$. A *natural transformation* from F to G is an assignment α of an arrow $\alpha(c) : Fc \to Gc$ in $\mathbf{D}$ (usually written α_c) to each $c \in \mathbf{C}_0$ with the property that for each arrow $f: a \to b$ of $\mathbf{C}$,

$$(*) \qquad \alpha_b \circ Ff = Gf \circ \alpha_a .$$

Complexes of arrows are often represented by diagrams. Thus, the equation (*) is equivalent to the statement that the diagram

[1]Determined by foundational difficulties: a category of all categories is rather like a set of all sets.

$$\begin{tikzcd} Fa \arrow[r, "Ff"] \arrow[d, "\alpha_a"'] & Fb \arrow[d, "\alpha_b"] \\ Ga \arrow[r, "Gf"'] & Gb \end{tikzcd} \qquad \text{(A.2)}$$

commutes (or *is commutative*), that is, the various maximal paths represent the same arrow.

A2. Example. Let $\sigma = \{S, f{:}S \to S\}$ be a signature. It determines a monoid $\mathbf{M}_\sigma$ with object S and arrow set $\{f\}^* = \{1, f, ff, fff, \ldots\}$. A σ-algebra **A** determines a functor $F_\mathbf{A}$: $\mathbf{M}_\sigma \to \mathbf{Set}$, with $F_\mathbf{A}(S) = \mathbf{A}(S)$ and $F_\mathbf{A}(f) = \mathbf{A}(f)$. Suppose **B** is another σ-algebra; then a natural transformation from $F_\mathbf{A}$ to $F_\mathbf{B}$ is precisely a homomorphism from **A** to **B**. (*Any* signature may be treated in this way, but complications arise almost instantly.)

A short notation for 'α is a natural transformation from F to G' is:

$$\alpha : F \overset{\bullet}{\to} G.$$

Natural transformations may be regarded as arrows of some category in two distinct ways.

a) The objects are functors, the domain of $\alpha : F \overset{\bullet}{\to} G$ is F and the codomain G. So if we think of a natural transformation as an arrow in this category, we may use the ordinary →-notation. The composite

$$F \xrightarrow{\alpha} G \xrightarrow{\beta} H$$

is defined by $(\beta \bullet \alpha)(c) = \beta_c \circ \alpha_c$. An identity 1_F (usually written F) assigns to every object c in dom$F$ the identity arrow 1_{Fc} $(= F(1_c))$. The category of all functors from **C** to **D** as obtained by this definition is denoted by $\mathbf{D}^\mathbf{C}$.

b) The objects are categories, if F and G are functors from **C** to **D** the domain of $\alpha : F \overset{\bullet}{\to} G$ is **C** and the codomain **D**, and if $\gamma : \mathbf{D} \to \mathbf{E}$ is a natural transformation from K to L, the composite $\gamma \circ \alpha$ is the natural transformation from KF to LG (with domain **C** and codomain **E**) defined by $(\gamma \circ \alpha)(c) = \gamma_{Gc} \circ K(\alpha_c)$ (or equivalently: $(\gamma \circ \alpha)(c) = L(\alpha_c) \circ \gamma_{Fc}$). The identity arrows are the identical transformations of the identity functors; thus $\mathrm{id}_\mathbf{C}$ assigns 1_c to every $c \in \mathbf{C}_0$, and this assignment is a natural transformation $1_\mathbf{C} \overset{\bullet}{\to} 1_\mathbf{C}$. (So $\mathrm{id}_\mathbf{C}$ in this category is $\mathrm{id}(\mathrm{id}_\mathbf{C})$ in the previous one. Since we are not concerned with the former $\mathrm{id}_\mathbf{C}$ outside this digression, there is no need for an unambiguous notation.)

The two compositions • and ∘ are related by the *exchange law*:

$$(\alpha\bullet\beta)\circ(\gamma\bullet\delta) = (\alpha\circ\gamma)\bullet(\beta\circ\delta).$$

The identity transformation $1_F : F \stackrel{\bullet}{\to} F$ will usually be written F. The exchange law implies among other things that $(\alpha\bullet\beta)\circ F = (\alpha\bullet\beta)\circ(F\bullet F) = (\alpha\circ F)\bullet(\beta\circ F)$.

An *initial object* in a category **C** is an object i of **C** such that for any object c of **C**, there is exactly one arrow $i \to c$ in **C**. It is immediate that any two initial objects in **C** are isomorphic.

Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. A *universal arrow* from $d \in \mathbf{D}_0$ to F is given by an object c of **C** and an arrow $g : d \to Fc$ with the property that for any c' in **C** and $h : d \to Fc'$, there exists precisely one $k : c \to c'$ in **C** such that $h = Fk \circ g$.

$$\begin{tikzcd}
d \arrow[r, "g"] \arrow[dr, "h"'] & Fc \arrow[d, "Fk"] & c \arrow[d] \\
& Fc' & c'
\end{tikzcd}$$

(A.2)

A wellknown example of a universal arrow is the canonical injection of the generators into a free algebra. In this case c is the free algebra, and F the 'forgetful' functor that reduces an algebra (of the signature of c) to its carrier set. The universal property is: given any algebra c', any mapping of the set d into the carrier set of c' has a unique extension to a homomorphism from c to c'. A related example occurs in §5; where F is the embedding of a category of $\boldsymbol{K}$-generated algebras into the category **Alg** of all algebras, d is an algebra, c is $F_{\boldsymbol{K}}(d)$, and $g = \eta_{\boldsymbol{K},d}$.

Universal arrows may be defined as initial objects in a suitable category. Let the *objects* of $d{\downarrow}F$ be the pairs $\langle c, f\rangle$ of an object c of **C** and an arrow f in **D** from d to Fc; the arrows from $\langle c, f\rangle$ to $\langle c, g\rangle$ the arrows $h : c \to c'$ in **C** such that $g = Fh\circ f$; and take the composition from **C**.

$$\begin{tikzcd}
& d \arrow[dl, "f"'] \arrow[dr, "g"] & \\
Fc \arrow[rr, "Fh"'] & & Fc'
\end{tikzcd}$$

(A.3)

Then a universal arrow from d to F is the same as an initial object in $d{\downarrow}F$.

Let $F: \mathbf{C} \to \mathbf{D}$ be a functor, $d \in \mathbf{D}_0$. A *cone from F to d* is a natural transformation $\alpha: F \xrightarrow{\bullet} K_d$, where K_d is the constant functor $c \mapsto d$, $f \mapsto 1_d$.[1] Since the K does not contribute to the message, I commonly write $\alpha: F \xrightarrow{\bullet} d$. Now it is easy to make K a functor from $\mathbf{D}$ to $\mathbf{D}^{\mathbf{C}}$: for $f: d \to d'$, set $K_f(c) = f$. A *colimit cone* for F is a universal arrow from F to K. That is to say, if $\kappa: F \xrightarrow{\bullet} d$ is a colimit cone, and $\alpha: F \xrightarrow{\bullet} e$ is any cocone, there is precisely one arrow $g: d \to e$ in $\mathbf{D}$ such that $\alpha = K_g \bullet \kappa$ (or shorter: $\alpha = g \bullet \kappa$).[2] The vertex d is called a *colimit* of F; it is easy to see that any two colimits of F are isomorphic.

A.3 Examples

(i) Let $\mathbf{C}$ be a category such that $\mathbf{C}_0$ and $\mathbf{C}_1$ are sets, and $F: \mathbf{C} \to \mathbf{Set}$ a functor. Put

$$D = \bigcup_{c \in \mathbf{C}_0} (Fc \times \{c\}).$$

Let Θ be the least equivalence relation on D such that for all $f \in \mathbf{C}_1$ and $x \in \mathrm{dom} Ff$,

$$\langle Ff(x), \mathrm{cod} f \rangle \equiv \langle x, \mathrm{dom} f \rangle.$$

We denote the equivalence class of $\langle x, c \rangle$ by $\kappa_c(x)$. Then κ is a cone from F to D; indeed, it is not hard to see that κ is a colimit cone.

(ii) The same construction may be carried through for I-sorted sets, for a given index set I; and a fortiori for algebras, except that we do not know how to define the required operations on D. For instance, the procedure of (i) does not guarantee that the interpretation of a constant symbol $\mathbf{k}$ in some Fc_1 will be made equivalent to the interpretation of $\mathbf{k}$ in Fc_2. The difficulty vanishes in case $\mathbf{C}$ is a *directed quasi-ordering* — i.e. a quasi-ordering such that for any c_1 and c_2, there exist $c_0 \succ c_1, c_2$. Then surely

$$\langle k_1, c_1 \rangle \equiv \langle k_0, c_0 \rangle \equiv \langle k_2, c_2 \rangle,$$

which fixes the interpretation of $\mathbf{k}$ in D; it is a simple matter to generalize this consideration to arbitrary operations.

A *coproduct* of two objects c and d in a category $\mathbf{C}$ is a colimit of the functor $F\ : 0 \mapsto c,\ 1 \mapsto d$ from the category 2 (with objects 0 and 1 and only identity arrows) to $\mathbf{C}$. It is customary to denote a favourite coproduct of c and d by $c+d$. Let κ be the colimit cone; $\kappa_0 : c \to c+d$ and $\kappa_1 : d \to c+d$ are called the *injections* of the coproduct. Suppose we are given arrows $f: c \to a$ and $g: d \to a$, repre-

[1] In fact I usually speak of *cocones*, in deference to the dual notion of a cone from d to F.
[2] It is evident that *a fortiori* colimit cones for F may be defined as initial objects in a category of cocones from F.

senting a cone from F to a. Then the unique arrow $h : c+d \to a$ such that $h \bullet \kappa_0 = f$ and $h \bullet \kappa_1 = g$ is denoted by $[f, g]$. By the properties of colimits, $c+d \cong d+c$ and $b+(c+d) \cong (b+c)+d$. The dual notion is that of a *product* $c \times d$, with projections $\pi_0 : c \times d \to c$ and $\pi_1 : c \times d \to d$, and for given $f : a \to c$ and $g : a \to d$ a unique $(f, g) : a \to c \times d$ such that $\pi_0 \circ (f, g) = f$ and $\pi_1 \circ (f, g) = g$. Generally, a (co-)product is any (co-)limit of a functor from a category with only identity arrows.

A.4 Examples

a) The product in **Set** is the wellknown Cartesian product. If $f_i : Y \to X_i$ $(1 \le i \le n)$ are functions, then $(f_1, \ldots, f_n)$ is the function f: $Y \to X_1 \times \ldots \times X_n$ defined by $f(y) = \langle f_1(y), \ldots, f_n(y) \rangle$.

b) Products in **Alg** may be based on products in **Set**. The operations in a product are defined ‘componentwise’; for example,

$$(*) \qquad (\mathbf{A} \times \mathbf{B})(\mathbf{f})(\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle) = \langle \mathbf{A}(\mathbf{f})(a_1, a_2), \mathbf{B}(\mathbf{f})(b_1, b_2) \rangle.$$

c) Products in categories of partial algebras are obtained analogously, with the obvious proviso that the lefthand side of (*) can be calculated if and only if the righthand side can. In particular, in a product of two categories the composite $\langle f_1, g_1 \rangle \circ \langle f_2, g_2 \rangle$ is defined, and then defined as $\langle f_1 \circ f_2, g_1 \circ g_2 \rangle$ whenever the latter makes sense.

We end by stating the dual of a theorem from [MaL, V§3]. Let **J**, **P** and **X** be categories. For any object p in **P** we define the evaluation functor $E_p : \mathbf{X}^{\mathbf{P}} \to \mathbf{X}$ by $E_p(H) = H(p)$ and $E_p(\alpha) = \alpha_p$.

A.5 Proposition. Suppose that $S : \mathbf{J} \to \mathbf{X}^{\mathbf{P}}$ is a functor with the property that for each $p \in \mathbf{P}_0$ the composite $E_p \circ S : \mathbf{J} \to \mathbf{X}$ has a colimit $\overline{p}$ with colimit cone κ_p. Then there is precisely one functor $K : \mathbf{P} \to \mathbf{X}$ such that for all $p \in \mathbf{P}_0$,
(i) $K(p) = \overline{p}$;
(ii) the assignment $p \mapsto \kappa_p$ is a cone κ from S to the object K of $\mathbf{X}^{\mathbf{P}}$.
Moreover, κ is a colimit cone for S.

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