A propositional logic with 4 values: true, false, divergent and meaningless

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Abstract

In this paper, we survey 3-valued logics and their complete axiomatizations, one of which is new. We then propose a 4-valued, functionally complete logic that incorporates these 3-valued systems and provide notations for interesting operators and subsystems.

Key Words & Phrases: 3-valued logic, 4-valued logic, axiomatics of non-classical theories.

1991 Mathematics Subject Classification: 03B50, 03B60, 03G10.

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1 Introduction

Three-valued logic is relevant from different points of view and has been studied intensively by many authors including D.A. Bochvar, S.C. Kleene and J. McCarthy. On the basis of an inspection of their systems, we adopt a couple of classical principles which exclude some prominent options such as the systems of A. Heyting, J. Łukasiewicz and N.D. Belnap. We thus obtain a survey of 3-valued logics with complete axiom systems, one of which is new. It turns out that in the various calculi the third, non-classical truth-value is associated with different intuitions, and that the systems considered can be embedded in a direct way in a single four-valued, functionally complete system with the values *true, false, divergent* and *meaningless.*

2 Three-valued propositional calculi

One way of formalizing partial predicates is to assume that they are total with values in the three-element set $\mathbb{T}_3 = \{t, f, *\}$ where $*$ acts as a marker indicating that an expression to which it is assigned is neither true nor false. Elementary school arithmetic tells us that, if we wish to design a three-valued system with the additional connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (implication) which extends the classical bivalent system, then there are 3^{16} possible compositional ways of doing so. Although most of these systems do not occur in the literature - as they lack an interesting interpretation of $*$ - one is confronted with a vast field with hundreds of published papers. We have attempted to keep this inventory short by focussing directly on the class of calculi which might be appropriate in our case. This class meets five conditions which are inspired by traditional classical logic.

The first condition concerns the *interdefinability of the propositional connectives.* We require that the system can be described completely by the behaviour of $\neg$ and $\wedge$, and that the remaining connectives can be defined in terms of these in the usual way by

(C1) $\quad b \vee b' = \neg(\neg b \wedge \neg b'), \quad b \rightarrow b' = \neg b \vee b'$, for all $b, b' \in \mathbb{T}_3$.

This first requirement reduces the number of possible calculi to 3^6. That is, we only consider systems which are determined by a completion of the following two incomplete tables:

$\neg$	
t	f
f	t
$*$	

$\wedge$	t	f	$*$
t	t	f	
f	f	f	
$*$			

The interdefinability condition is a very restrictive one and excludes prominent calculi such as those due to J. Łukasiewicz [L20] and A. Heyting [H30].

$\neg$	
t	f
f	t
$*$	$*$

$\wedge$	t	f	$*$
t	t	f	$*$
f	f	f	f
$*$	$*$	f	$*$

$\vee$	t	f	$*$
t	t	t	t
f	t	f	$*$
$*$	t	$*$	$*$

$\rightarrow$	t	f	$*$
t	t	f	$*$
f	t	t	t
$*$	t	$*$	t

Table 1: Łukasiewicz's three-valued system.

$\neg$	
t	f
f	t
$*$	f

$\wedge$	t	f	$*$
t	t	f	$*$
f	f	f	f
$*$	$*$	f	$*$

$\vee$	t	f	$*$
t	t	t	t
f	t	f	$*$
$*$	t	$*$	$*$

$\rightarrow$	t	f	$*$
t	t	f	$*$
f	t	t	t
$*$	t	f	t

Table 2: Heyting's three-valued system.

In Łukasiewicz's system $*$ is to be read as *possible* and the intent is that future contingent propositions are the ones which take that value. There is in fact only one value that spoils interdefinability of the connectives, namely the one for $* \rightarrow *$. Unfortunately, Łukasiewicz was not very explicit on this crucial point, but it is very likely that he designed his system in this way because he wanted $\phi \rightarrow \phi$ to be a three-valued tautology. Heyting originally devised his system to show that $\neg\neg\phi \rightarrow \phi$ is not a theorem of intuitionistic logic and in doing so he necessarily had to come up with a system in which the traditional definability results fail.

The second, third and fourth condition extend the *double negation principle*, the *idempotency of* $\wedge$ and the principle of t *being a unit for* $\wedge$ to $*$. That is, we require that the system meets the following additional three conditions:

(C2) $\neg\neg * = *$,

(C3) $* \wedge * = *$,

(C4) $* \wedge t = * = t \wedge *$.

These requirements further reduce the number of possible calculi. We are left with 9 systems which are determined by negation and a completion of the still incomplete table for $\wedge$:

$\neg$	
t	f
f	t
$*$	$*$

$\wedge$	t	f	$*$
t	t	f	$*$
f	f	f	
$*$	$*$		$*$

We do not know of any interesting system with interdefinable connectives that does not adhere to (C2) and (C3). Belnap [Bp70], however, proposed a three-valued system with $\neg$, $\wedge$ and $\vee$ that meets (C1)-(C3) but rejects (C4):

$\neg$	
t	f
f	t
$*$	$*$

$\wedge$	t	f	$*$
t	t	f	t
f	f	f	f
$*$	t	f	$*$

$\vee$	t	f	$*$
t	t	t	t
f	t	f	f
$*$	t	f	$*$

Table 3: Belnap's three-valued system.

In Belnap's system $*$ is to be read as *unassertive*. A conjunction or disjunction takes that value only in case it consists of two unassertive components. In the remaining cases the value is determined by either traditional classical logic or the assertive component. However, as Belnap observes himself, it is unclear whether - given this interpretation of $*$ - the system can be enriched with an appropriate implication.

The last requirement,

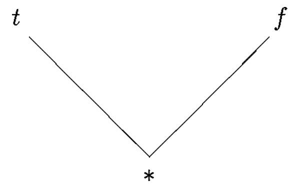
(C5) $* \wedge f \neq t \neq f \wedge *$,

seems to us only a sensible one, and reduces the number of possible calculi to the four tabled below:

$\neg$	
t	f
f	t
$*$	$*$

$\wedge$	t	f	$*$
t	t	f	$*$
f	f	f	$f/*$
$*$	$*$	$f/*$	$*$

These four remaining calculi, in which $*$ may be either read as *undefined* or as *incomplete*, are well-known from the literature. They are shown in Table 4. Note that the operators of these systems share the important monotonicity property with respect to the partial ordering on $\mathbb{T}_3$ defined by:



A unary operator $g : \mathbb{T}_3 \to \mathbb{T}_3$ is called *monotone* if $b \leq b'$ implies $g(b) \leq g(b')$. In our case this is equivalent to the fact that

$$g(*) \leq g(t) \text{ and } g(*) \leq g(f).$$

A binary ore more-argument operator is monotone, if it is monotone in each argument separately with other arguments arbitrarily fixed.

The first of these calculi, also known as *strict three-valued logic*, is due to D.A. Bochvar [Bo39]. Here, on the theory that one bad apple spoils the barrel, an expression has the value $*$ as soon as it has a component with that value. This corresponds directly to the situation where instead of considering three-valued total predicates one is considering two-valued partial ones. However, as is frequently argued (cf. e.g. [Bl88]), this approach is not convenient in general in view of the following use of case constructors: Assume that we are given two total predicates $P : D \to \{t, f\}$ and $P' : D' \to \{t, f\}$. Assuming furthermore that D and D' are disjoint, P and P' can be extended to functions from $D \cup D'$ to $\mathbb{T}_3$ by $P(x) = *$, if $x \in D'$, and $P'(x) = *$, if $x \in D$. Assume now that on the ground of this calculus one wants to define a third predicate $P'' : (D \cup D') \to \{t, f\}$ such that $P'' \restriction D = P$ and $P'' \restriction D' = P'$, so P'' tests the property expressed by P in D and the property expressed by P' in D'. The straightforward attempt to define P'' by

$$P''(x) = (x \in D \to P(x)) \ \wedge \ (x \in D' \to P'(x))$$

fails, as for all $x \in D \cup D'$ either $P(x) = *$ or $P'(x) = *$. In fact, as is easy to show, P'' cannot be defined as a propositional combination of P and P', if one uses strict connectives.

The other three calculi are *non-strict* in the sense that a compound formula can have a classical truth-value even if some of its components are assigned the value $*$. Here one evaluates expressions *lazily*, that is, one assumes that all subexpressions of a given expression have defined values and only evaluates as many of them as is necessary to establish the value of the whole expression. Basically, there are two methods of lazy evaluation: the first (in historical order), in which evaluation is executed in parallel, was described in the context of computational mathematics by S.C. Kleene [K38]; the second, programmingoriented method, in which evaluation proceeds sequentially from left to right, was proposed by J. McCarthy [MC63]. McCarthy's calculus has of course a dual in which the order of evaluation is reversed.

Strict three-valued logic and the calculus of Kleene inherit many nice algebraic properties of traditional classical logic; the calculi of McCarthy, are a little less well-off in this respect. For instance,

$\neg$	
t	f
f	t
$*$	$*$

$\wedge$	t	f	$*$
t	t	f	$*$
f	f	f	$*$
$*$	$*$	$*$	$*$

$\vee$	t	f	$*$
t	t	t	$*$
f	t	f	$*$
$*$	$*$	$*$	$*$

$\rightarrow$	t	f	$*$
t	t	f	$*$
f	t	t	$*$
$*$	$*$	$*$	$*$

Strict three-valued logic $\mathbb{S}_3$

$\neg$	
t	f
f	t
$*$	$*$

$\wedge$	t	f	$*$
t	t	f	$*$
f	f	f	f
$*$	$*$	f	$*$

$\vee$	t	f	$*$
t	t	t	t
f	t	f	$*$
$*$	t	$*$	$*$

$\rightarrow$	t	f	$*$
t	t	f	$*$
f	t	t	t
$*$	t	$*$	$*$

Kleene's three-valued logic $\mathbb{K}_3$

$\neg$	
t	f
f	t
$*$	$*$

$\wedge$	t	f	$*$
t	t	f	$*$
f	f	f	f
$*$	$*$	$*$	$*$

$\vee$	t	f	$*$
t	t	t	t
f	t	f	$*$
$*$	$*$	$*$	$*$

$\rightarrow$	t	f	$*$
t	t	f	$*$
f	t	t	t
$*$	$*$	$*$	$*$

McCarthy's three-valued logic $\mathbb{M}_3$

$\neg$	
t	f
f	t
$*$	$*$

$\wedge$	t	f	$*$
t	t	f	$*$
f	f	f	$*$
$*$	$*$	f	$*$

$\vee$	t	f	$*$
t	t	t	$*$
f	t	f	$*$
$*$	t	$*$	$*$

$\rightarrow$	t	f	$*$
t	t	f	$*$
f	t	t	$*$
$*$	t	$*$	$*$

McCarthy's dual $\mathbb{Md}_3$

Table 4: The three-valued logics $\mathbb{S}_3$, $\mathbb{K}_3$, $\mathbb{M}_3$ and $\mathbb{Md}_3$.

- $\wedge$, $\vee$ are commutative ($\mathbb{S}_3$, $\mathbb{K}_3$),
- $\wedge$ is right distributive over $\vee$ ($\mathbb{S}_3$, $\mathbb{K}_3$, $\mathbb{M}\mathrm{d}_3$),
- $\vee$ is right distributive over $\wedge$ ($\mathbb{S}_3$, $\mathbb{K}_3$, $\mathbb{M}\mathrm{d}_3$),
- $\vee$ is left distributive over $\wedge$ ($\mathbb{S}_3$, $\mathbb{K}_3$, $\mathbb{M}_3$),
- $\wedge$ is left distributive over $\vee$ ($\mathbb{S}_3$, $\mathbb{K}_3$, $\mathbb{M}_3$),
- $\wedge$, $\vee$ is associative ($\mathbb{S}_3$, $\mathbb{K}_3$, $\mathbb{M}_3$, $\mathbb{M}\mathrm{d}_3$),
- the laws of de Morgan hold ($\mathbb{S}_3$, $\mathbb{K}_3$, $\mathbb{M}_3$, $\mathbb{M}\mathrm{d}_3$).

Of course, there are also classical tautologies, such as *tertium non datur*, which do not hold in any of the calculi. The axiomatizations of $\mathbb{S}_3$, $\mathbb{K}_3$, $\mathbb{M}_3$ and $\mathbb{M}\mathrm{d}_3$, which define these systems completely, are given in Table 5 and Table 6. The completeness of (K1)–(K10) follows from [Ka58], the completeness of (M1)–(M11) and (Md1)–(Md11) is proved in [GS90]. A completeness proof of (S1)–(S11) is included in Appendix A.

(S1)	$\neg t$	$=$	f
(S2)	$\neg *$	$=$	$*$
(S3)	$\neg\neg x$	$=$	x
(S4)	$\neg(x \wedge y)$	$=$	$\neg x \vee \neg y$
(S5)	$x \rightarrow y$	$=$	$\neg x \vee y$
(S6)	$x \wedge (y \wedge z)$	$=$	$(x \wedge y) \wedge z$
(S7)	$t \wedge x$	$=$	x
(S8)	$x \vee (\neg x \wedge y)$	$=$	$x \vee y$
(S9)	$x \wedge y$	$=$	$y \wedge x$
(S10)	$x \wedge (y \vee z)$	$=$	$(x \wedge y) \vee (x \wedge z)$
(S11)	$* \wedge x$	$=$	$*$
(K1)	$\neg t$	$=$	f
(K2)	$\neg *$	$=$	$*$
(K3)	$\neg\neg x$	$=$	x
(K4)	$\neg(x \wedge y)$	$=$	$\neg x \vee \neg y$
(K5)	$x \rightarrow y$	$=$	$\neg x \vee y$
(K6)	$x \wedge (y \wedge z)$	$=$	$(x \wedge y) \wedge z$
(K7)	$t \wedge x$	$=$	x
(K8)	$x \vee (x \wedge y)$	$=$	x
(K9)	$x \wedge y$	$=$	$y \wedge x$
(K10)	$x \wedge (y \vee z)$	$=$	$(x \wedge y) \vee (x \wedge z)$

Table 5: The axiomatizations of $\mathbb{S}_3$ and $\mathbb{K}_3$.

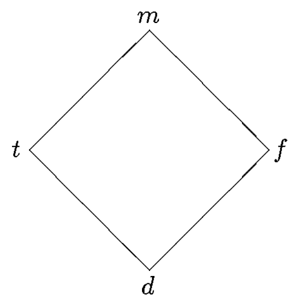
$$
\begin{array}{lrcl}
\text{(M1)} & \neg t & = & f \\
\text{(M2)} & \neg * & = & * \\
\text{(M3)} & \neg\neg x & = & x \\
\text{(M4)} & \neg(x \wedge y) & = & \neg x \vee \neg y \\
\text{(M5)} & x \rightarrow y & = & \neg x \vee y \\
\text{(M6)} & x \wedge (y \wedge z) & = & (x \wedge y) \wedge z \\
\text{(M7)} & t \wedge x & = & x \\
\text{(M8)} & x \vee (x \wedge y) & = & x \\
\text{(M9)} & x \wedge (y \vee z) & = & (x \wedge y) \vee (x \wedge z) \\
\text{(M10)} & (x \vee y) \wedge z & = & (x \wedge z) \vee (\neg x \wedge y \wedge z) \\
\text{(M11)} & (x \wedge y) \vee (y \wedge x) & = & (y \wedge x) \vee (x \wedge y) \\
\\
\text{(Md1)} & \neg t & = & f \\
\text{(Md2)} & \neg * & = & * \\
\text{(Md3)} & \neg\neg x & = & x \\
\text{(Md4)} & \neg(x \wedge y) & = & \neg x \vee \neg y \\
\text{(Md5)} & x \rightarrow y & = & \neg x \vee y \\
\text{(Md6)} & x \wedge (y \wedge z) & = & (x \wedge y) \wedge z \\
\text{(Md7)} & t \wedge x & = & x \\
\text{(Md8)} & (x \wedge y) \vee y & = & y \\
\text{(Md9)} & (x \vee y) \wedge z & = & (x \wedge z) \vee (y \wedge z) \\
\text{(Md10)} & x \wedge (y \vee z) & = & (x \wedge y \wedge \neg z) \vee (x \wedge z) \\
\text{(Md11)} & (x \wedge y) \vee (y \wedge x) & = & (y \wedge x) \vee (x \wedge y)
\end{array}
$$

Table 6: The axiomatizations of $\mathbb{M}_3$ and $\mathbb{M}\mathrm{d}_3$.

3 Incorporating three-valued logic into four-valued logic

In this section we propose a four-valued logic that accommodates strict, parallel and sequential evaluation.

$\mathbb{T}_4$, the set of values of this logic, consists of t, f, d (divergent) and m (meaningless) and is structured by the following partial ordering reflecting information content:



The set of logical operators consists of negation ($\neg$), a unary *definedness* operator ($\downarrow$), a binary *strict parallel conjunction* ($\wedge$), that combines strict three-valued conjunction with respect to m and Kleene's three-valued conjunction with respect to d, and a binary *left first sequential conjunction* (${}_{\circ}\!\wedge$), that adapts McCarthy's conjunction to the domain of four truth values. $\neg$, $\wedge$ and ${}_{\circ}\!\wedge$ are monotone with respect to the ordering given above; $\downarrow$ is non-monotone, since e.g. $t \le m$ but $\downarrow t \not\le \downarrow m$. We shall call this logic $\Sigma_4(\neg, \wedge, \downarrow, {}_{\circ}\!\wedge)$.

$\neg$	
m	m
t	f
f	t
d	d

$\downarrow$	
m	f
t	t
f	t
d	f

$\wedge$	m	t	f	d
m	m	m	m	m
t	m	t	f	d
f	m	f	f	f
d	m	d	f	d

${}_{\circ}\!\wedge$	m	t	f	d
m	m	m	m	m
t	m	t	f	d
f	f	f	f	f
d	d	d	d	d

Table 7: The connectives of $\Sigma_4(\neg, \wedge, \downarrow, {}_{\circ}\!\wedge)$.

Suppose we add to $\Sigma_4(\neg, \wedge, \downarrow, {}_{\circ}\!\wedge)$ a new n-ary connective γ which we wish to interpret as a truth function. Then we must specify its interpretation by a truth table which for every sequence of n m's, t's, f's and d's assigns either m, t, f or d. If we do this, we have added a *truth-functional connective* to $\Sigma_4(\neg, \wedge, \downarrow, {}_{\circ}\!\wedge)$. We shall show that every truth-functional connective can be defined in terms of m, t, d, $\neg$, $\downarrow$, $\wedge$ and ${}_{\circ}\!\wedge$ in $\Sigma_4(\neg, \wedge, \downarrow, {}_{\circ}\!\wedge)$, that is, $\{m, t, d, \neg, \downarrow, \wedge, {}_{\circ}\!\wedge\}$ is *truth-functionally complete*. More precisely, given a connective γ as above, we can find a formula $\phi(x_1, \ldots, x_n)$ which uses only $\{x_1, \ldots, x_n, m, t, d, \neg, \downarrow, \wedge, {}_{\circ}\!\wedge\}$, such that for every sequence $a_1, \ldots, a_n$ of m's, t's, f's and d's we have that $\phi(a_1, \ldots, a_n) = \gamma(a_1, \ldots, a_n)$. Thus except for the convenience of abbreviation, further truth-functional connectives add nothing to the expressivity of $\Sigma_4(\neg, \wedge, \downarrow, {}_{\circ}\!\wedge)$.

Theorem 3.1. *$\Sigma_4(\neg, \wedge, \downarrow, {}_{\circ}\!\wedge)$ is truth-functionally complete.*

Proof. Let γ be an n-ary truth-functional connective as above. As basic building blocks of its $\Sigma_4(\neg, \wedge, \downarrow, {}_{\circ}\!\wedge)$-representation we take

$$\kappa_m(x) \triangleq \neg \downarrow (\downarrow x \ \wedge \ x),$$

$$\kappa_t(x) \triangleq \downarrow x \mathbin{{}_{\circ}\!\wedge} x,$$

$$\kappa_f(x) \triangleq \kappa_t(\neg x),$$

$$\kappa_d(x) \triangleq \neg \downarrow (\neg\kappa_e(x) \mathbin{{}_{\circ}\!\wedge} x).$$

Observe that for all $a, a' \in \mathbb{T}_4$, we have that

$$\kappa_a(a') = \begin{cases} t & \text{if } a = a' \\ f & \text{otherwise.} \end{cases}$$

Next we put

$$\phi_{a_1,\ldots,a_n}(x_1,\ldots,x_n) \triangleq (\kappa_{a_1}(x_1) \ \wedge\ \cdots\ \wedge\ \kappa_{a_n}(x_n)) \mathbin{{}_{\circ}\!\wedge} \hat{\gamma}(a_1,\ldots,a_n)$$

where $\hat{\gamma}(a_1,\ldots,a_n) \triangleq \neg t$ if $\gamma(a_1,\ldots,a_n) = f$, and $\hat{\gamma}(a_1,\ldots,a_n) \triangleq \gamma(a_1,\ldots,a_n)$, otherwise. Then

$$\phi_{a_1,\ldots,a_n}(a'_1,\ldots,a'_n) = \begin{cases} \gamma(a_1,\ldots,a_n) & \text{if for all } 1 \le i \le n,\ a_i = a'_i \\ f & \text{otherwise,} \end{cases}$$

for all $a_1,\ldots,a_n,a'_1,\ldots,a'_n \in \mathbb{T}_4$. Finally we define

$$\phi(x_1,\ldots,x_n) \triangleq \neg(\bigwedge\{\neg\phi_{a_1,\ldots,a_n}(x_1,\ldots,x_n) \mid a_1,\ldots,a_n \in \mathbb{T}_4\})$$

where $\bigwedge$ indicates the strict parallel conjunction over all the elements in the set, associating e.g. to the right. Then $\phi(x_1,\ldots,x_n)$ is an $\Sigma_4(\neg,\wedge,\downarrow,{}_{\circ}\!\wedge)$-representation of γ. □

In [BH89] it is proved that every finite algebra can be axiomatized completely with the aid of auxiliary function symbols. Since $\Sigma_4(\neg,\wedge,\downarrow,{}_{\circ}\!\wedge)$ is truth-functionally complete, this means that a complete set of axioms exists for it. We are still working on a (more interesting) direct axiomatization.

Being truth-functionally complete, $\Sigma_4(\neg,\wedge,\downarrow,{}_{\circ}\!\wedge)$ describes the whole function space over $\mathbb{T}_4$. However, as they are interesting in their own right, we shall introduce a few additional connectives. First of all, there are the disjunctive versions of the conjunctions considered so far:

$$x \ \vee\ y \triangleq \neg(\neg x \ \wedge\ \neg y),$$

$$x \mathbin{{}^{\circ}\!\vee} y \triangleq \neg(\neg x \mathbin{{}_{\circ}\!\wedge} \neg y),$$

and the sequential con- and disjunction with reversed order of evaluation:

$$x \mathbin{\wedge_{\circ}} y \triangleq y \mathbin{{}_{\circ}\!\wedge} x,$$

$$x \mathbin{\vee^{\circ}} y \triangleq \neg(\neg x \mathbin{\wedge_{\circ}} x).$$

We shall call these connectives *strict parallel disjunction*, *left first sequential disjunction*, *right first sequential conjunction* and *right first sequential disjunction*, respectively. Next, there is a conjunction and a disjunction which are strict in both m and d while giving preference to m over d:

$$x \mathbin{{}_{\circ}\!\wedge_{\circ}} y \triangleq (x \mathbin{{}_{\circ}\!\wedge} y) \ \vee\ (x \mathbin{\wedge_{\circ}} y),$$

$$x \mathbin{{}^{\circ}\!\vee^{\circ}} y \triangleq \neg(\neg x \mathbin{{}_{\circ}\!\wedge_{\circ}} \neg y).$$

We shall call these connectives *double strict conjunction* and *double strict disjunction*, respectively. Finally, there is a unary *significance* operator that is strict in m and constant t on the remaining truth values:

$$\Downarrow x \triangleq \downarrow\downarrow x \ \vee\ x.$$

The truth tables for the operators $\wedge_{\circ}$, ${}_{\circ}\!\wedge_{\circ}$ and $\Downarrow$ are given in Table 8.

$\wedge_{\circ}$	m	t	f	d
m	m	m	f	d
t	m	t	f	d
f	m	f	f	d
d	m	d	f	d

${}_{\circ}\!\wedge_{\circ}$	m	t	f	d
m	m	m	m	m
t	m	t	f	d
f	m	f	f	d
d	m	d	d	d

$\Downarrow$	
m	m
t	t
f	t
d	t

Table 8: The connectives $\wedge_{\circ}$, ${}_{\circ}\!\wedge_{\circ}$ and $\Downarrow$.

There are several subsystems of $\Sigma_4(\neg, \wedge, \downarrow, {}_{\circ}\!\wedge)$ which are not as powerful but still correspond to interesting classes of functions. We end this section by mentioning some more obvious ones. The nomenclature for these systems follows that one for the main system: that is, they are labelled $\Sigma_4(o_1, \ldots, o_n)$ where $o_1, \ldots, o_n$ lists the primitive operators the system has in addition to the four constants. Except for the first system, complete axiomatizations are not known to exist.

$\Sigma_4(\neg, {}_{\circ}\!\wedge_{\circ})$: The system of the double strict extensions of classical functions. The axiomatization of this system is given in Table 9. Completeness is proved in Appendix B.

$$
\begin{array}{lrcl}
\text{(D1)} & \neg t & = & f \\
\text{(D2)} & \neg m & = & m \\
\text{(D3)} & \neg d & = & d \\
\text{(D4)} & \neg\neg x & = & x \\
\text{(D5)} & \neg(x \,{}_{\circ}\!\wedge_{\circ}\, y) & = & \neg x \,{}^{\circ}\!\vee^{\circ}\, \neg y \\
\text{(D6)} & x \,{}_{\circ}\!\wedge_{\circ}\, (y \,{}_{\circ}\!\wedge_{\circ}\, z) & = & (x \,{}_{\circ}\!\wedge_{\circ}\, y) \,{}_{\circ}\!\wedge_{\circ}\, z \\
\text{(D7)} & t \,{}_{\circ}\!\wedge_{\circ}\, x & = & x \\
\text{(D8)} & x \,{}^{\circ}\!\vee^{\circ}\, (\neg x \,{}_{\circ}\!\wedge_{\circ}\, y) & = & x \,{}^{\circ}\!\vee^{\circ}\, y \\
\text{(D9)} & x \,{}_{\circ}\!\wedge_{\circ}\, y & = & y \,{}_{\circ}\!\wedge_{\circ}\, x \\
\text{(D10)} & x \,{}_{\circ}\!\wedge_{\circ}\, (y \,{}^{\circ}\!\vee^{\circ}\, z) & = & (x \,{}_{\circ}\!\wedge_{\circ}\, y) \,{}^{\circ}\!\vee^{\circ}\, (x \,{}_{\circ}\!\wedge_{\circ}\, z) \\
\text{(D11)} & m \,{}_{\circ}\!\wedge_{\circ}\, x & = & m
\end{array}
$$

Table 9: The axiomatization of $\Sigma_4(\neg, {}_{\circ}\!\wedge_{\circ})$.

$\Sigma_4(\neg, {}_{\circ}\!\wedge)$: The system of the strongly sequential - not necessarily extended classical - functions.

$\Sigma_4(\neg, \Downarrow, {}_{\circ}\!\wedge, {}_{\circ}\!\wedge_{\circ})$: The system of the weakly sequential - not necessarily extended classical - functions.

$\Sigma_4(\neg, {}_{\circ}\!\wedge, \wedge)$: The system of the parallel - not necessarily extended classical - functions.

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Appendix A

We let $\mathcal{S}_3$ be the three-element algebra $\{t, f, *\}$ with operations $\neg, \wedge, \vee, \rightarrow$ defined by the $\mathbb{S}_3$-tables in Table 4. In this appendix we shall prove that (S1)–(S11) is a *complete* set of axioms. That is, for arbitrary open or closed terms s, s', we have that

$$\text{(S1)–(S11)} \vdash s = s' \text{ if and only if } \mathcal{S}_3 \models s = s'.$$

In order to prove completeness, we shall follow the proofs presented in [Ka58] and [GS90].

Given an arbitrary algebra $\mathcal{A}$ with carrier A, we let Δ_A denote the trivial congruence on A, i.e. $\Delta_A = \{(a, a) \mid a \in A\}$. $\mathcal{A}$ is called *subdirectly irreducible* if there is a smallest non-trivial congruence relation on A. By definition, an $\mathbb{S}_3$-*algebra* will be an algebra with elements t, f and $*$, and operations $\neg$, $\wedge$, $\vee$ and $\rightarrow$ which satisfies the laws (S1)–(S11) of Table 5. Clearly $\mathcal{S}_3$ is an $\mathbb{S}_3$-algebra. We shall show that $\mathcal{S}_3$ and the degenerated two-element algebra, which identifies t and f, are the only subdirectly irreducible $\mathbb{S}_3$-algebras. Completeness then follows from the famous Subdirect Representation Theorem of G. Birkhoff [Bi44].

We start with collecting a few properties of $\mathbb{S}_3$-algebras which will be used in the sequel. First, observe that the laws (S1), (S2), (S3) and (S4) imply that the variety of $\mathbb{S}_3$-algebras satisfies a duality principle similar to the one known from ordinary Boolean algebra. That is, if L is a law, then so is its dual L$'$ obtained from L by interchanging t and f, and $\wedge$ and $\vee$ throughout.

Lemma A.1. Every $\mathbb{S}_3$-algebra satisfies the following equations.

$$\begin{array}{lrcl}
\text{(S4)}' & \neg(x \vee y) & = & \neg x \wedge \neg y \\
\text{(S6)}' & x \vee (y \vee z) & = & (x \vee y) \vee z \\
\text{(S7)}' & f \vee x & = & x \\
\text{(S8)}' & x \wedge (\neg x \vee y) & = & x \wedge y \\
\text{(S9)}' & x \vee y & = & y \vee x \\
\text{(S10)}' & x \vee (y \wedge z) & = & (x \vee y) \wedge (x \vee z) \\
\text{(S11)}' & * \vee x & = & x
\end{array}$$

Proof. By a standard argument. □

It follows that the operation $\wedge$ is idempotent. The idempotency of $\vee$, which follows by duality, will not be needed in the remainder of the proof.

Lemma A.2. Every $\mathbb{S}_3$-algebra satisfies the following equation.

$$\text{(S12)} \quad x \wedge x \ = \ x.$$

Proof.

$$\begin{array}{rcll} x & = & x \vee f & (\mathrm{S7})', (\mathrm{S9})' \\ & = & x \vee (\neg x \wedge f) & (\mathrm{S8}) \\ & = & (x \vee \neg x) \wedge (x \vee f) & (\mathrm{S10})' \\ & = & (f \vee x) \wedge (\neg x \vee x) & (\mathrm{S9}), (\mathrm{S9})' \\ & = & x \wedge (\neg x \vee x) & (\mathrm{S7})' \\ & = & x \wedge x & (\mathrm{S8})' \end{array}$$

□

For later reference, we prove a property of t and f.

Lemma A.3. Let $\mathcal{A}$ be an $\mathbb{S}_3$-algebra.

a) Let $a, a' \in A$. If $a \wedge a' \in \{t, f\}$, then $a \wedge \neg a = f$.

b) If $* \in \{t, f\}$, then $card(A) = 1$.

Proof. (a) If $a \wedge a' = t$, then

$$\begin{array}{rcll} a & = & a \wedge t & (\mathrm{S7}), (\mathrm{S9}) \\ & = & a \wedge (a \wedge a') & \\ & = & (a \wedge a) \wedge a' & (\mathrm{S6}) \\ & = & a \wedge a' & (\mathrm{S12}) \\ & = & t. & \end{array}$$

So $a \wedge \neg a = f$. And if $a \wedge a' = f$, then

$$\begin{array}{rcll} f & = & a \wedge a' & \\ & = & a \wedge (\neg a \vee a') & (\mathrm{S8})' \\ & = & (a \wedge \neg a) \vee (a \wedge a') & (\mathrm{S10}) \\ & = & (a \wedge \neg a) \vee f & \\ & = & a \wedge \neg a & (\mathrm{S9})', (\mathrm{S7})'. \end{array}$$

(b) If $* = f$, then $* = t$ by (S1), (S2), (S3). And if $* = t$, then $a = t \wedge a = * \wedge a = *$, for each $a \in A$, by (S7), (S11). □

We shall now concentrate on congruences on $\mathbb{S}_3$-algebras which are crucial for Theorem A.6: given an $\mathbb{S}_3$-algebra $\mathcal{A}$ we define

- for $a \in A$, $\Theta_a = \{(a', a'') \in A \times A \,|\, a \vee a' = a \vee a''\}$;
- $\Gamma = A \backslash \{t, f\} \times A \backslash \{t, f\} \cup \{(t, t), (f, f)\}$.

Lemma A.4. Let $\mathcal{A}$ be an $\mathbb{S}_3$-algebra and $a \in A$. Then

a) Θ_a is a congruence relation on $\mathcal{A}$;

b) $\Theta_a \cap \Theta_{\neg a} \subseteq \Theta_{a \wedge \neg a}$;

c) $\Theta_a = \Delta_A$ iff $a = f$.

Proof. a) As $\vee$ and $\rightarrow$ are defined operations, it suffices to show that the substitution property holds for $\neg$ and $\wedge$. If $(a_1, a_2) \in \Theta_a$, it follows from S4$'$ that $\neg a \wedge \neg a_1 = \neg a \wedge \neg a_2$. Thus

$$\begin{array}{rcll} a \vee \neg a_1 & = & a \vee (\neg a \wedge \neg a_1) & \text{(S8)} \\ & = & a \vee (\neg a \wedge \neg a_2) & \\ & = & a \vee \neg a_2 & \text{(S8)} \end{array}$$

So $(\neg a_1, \neg a_2) \in \Theta_a$. Also, if $(a_1, a_2), (a_3, a_4) \in \Theta_a$, then

$$\begin{array}{rcll} a \vee (a_1 \wedge a_3) & = & (a \vee a_1) \wedge (a \vee a_3) & \text{(S10)}' \\ & = & (a \vee a_2) \wedge (a \vee a_4) & \\ & = & a \vee (a_2 \wedge a_4) & \text{(S10)}' \end{array}$$

Hence $(a_1 \wedge a_3, a_2 \wedge a_4) \in \Theta_a$.
b) If $(a_1, a_2) \in \Theta_a \cap \Theta_{\neg a}$, then

$$\begin{array}{rcll} (a \wedge \neg a) \vee a_1 & = & (a \vee a_1) \wedge (\neg a \vee a_1) & \text{(S9)}', \text{(S10)}' \\ & = & (a \vee a_2) \wedge (\neg a \vee a_2) & \\ & = & (a \wedge \neg a) \vee a_2 & \text{(S9)}', \text{(S10)}' \end{array}$$

Thus $(a_1, a_2) \in \Theta_{a \wedge \neg a}$.
c) If $\Theta_a = \Delta_A$, then $t = \neg a \wedge t$, since $(t, \neg a \wedge t) \in \Theta_a$ by (S8). Hence $\neg a = t$ by (S9), (S10), so that $a = f$ by (S1), (S3). The reverse follows from (S7)$'$. □

Lemma A.5. Let $\mathcal{A}$ be an $\mathbb{S}_3$-algebra. Suppose that for every $a \in A \backslash \{t, f\}$, $a \wedge \neg a \neq f$. Then Γ is a congruence relation on $\mathcal{A}$.

Proof. By Lemma A.3(a). □

Now let $\mathcal{S}_2$ denote the degenerated two-element $\mathbb{S}_3$-algebra which identifies t and f. That is, $\mathcal{S}_2 = \{\{\mathit{tf}, *\}, \neg, \wedge, \vee, \rightarrow\}$ where $\neg = id$ and $\wedge : \{\mathit{tf}, *\} \times \{\mathit{tf}, *\} \rightarrow \{\mathit{tf}, *\}$ is such that $x \wedge x = x$, $x \wedge y = y \wedge x$ and $x \wedge * = *$.

Theorem A.6. $\mathcal{S}_2$ and $\mathcal{S}_3$ are the only subdirectly irreducible $\mathbb{S}_3$-algebras.

Proof. Clearly, $\mathcal{S}_2$ and $\mathcal{S}_3$ are subdirectly irreducible. Assume that $\mathcal{A}$ is a subdirectly irreducible $\mathbb{S}_3$-algebra and that ϕ is the smallest non-trivial congruence relation on $\mathcal{A}$. Then

(i) $\phi \subseteq \Theta_a$, for each $a \in A \backslash \{t, f\}$: by Lemma A.4(c).

(ii) $a \wedge \neg a \neq f$, for each $a \in A \backslash \{t, f\}$: For, if $a \in A \backslash \{t, f\}$, then $\phi \subseteq \Theta_a \cap \Theta_{\neg a} \subseteq \Theta_{a \wedge \neg a}$ by (i) and Lemma A.4(b). Hence $a \wedge \neg a \neq f$ by Lemma A.4(c).

(iii) Γ is a congruence relation on $\mathcal{A}$: by (ii) and Lemma A.5.

(iv) $* \in A \backslash \{t, f\}$: For otherwise, $\mathcal{A}$ is the 1-element algebra by Lemma A.3(b) and therefore not subdirectly irreducible.

Now define $\Theta = \bigcap_{a \in A \backslash \{t, f\}} \Theta_a$. Then $\phi \subseteq \Theta$ by (i). However,

(v) $\Theta \cap \Gamma = \Delta_A$: For, let $(a, a') \in \Theta \cap \Gamma$ and assume that $a \neq a'$. Then $a, a' \in A \backslash \{t, f\}$ and $(a, a') \in \Theta_{\neg a} \cap \Theta_{\neg a'}$. Thus

$$\begin{array}{rcll} a & = & a \wedge a & \text{(S12)} \\ & = & a \wedge (\neg a \vee a) & \text{(S8)}' \\ & = & a \wedge (\neg a \vee a') & \text{since } (a, a') \in \Theta_{\neg a} \\ & = & a \wedge a' & \text{(S8)}' \end{array}$$

Similarly, $a' = a \wedge a'$. Hence $a = a'$, a contradiction.

It follows that $\Gamma = \Delta_A$, so that $A \backslash \{t, f\} = \{*\}$ by (iv). Therefore $\mathcal{A} = \mathcal{S}_2$ or $\mathcal{A} = \mathcal{S}_3$, as required. □

Corollary A.7. (S1)–(S11) completely axiomatize the equational theory of $\mathcal{S}_3$.

Proof. Assume $\mathcal{S}_3 \models s = s'$. Then $s = s'$ holds in any direct product of copies of $\mathcal{S}_2$ and $\mathcal{S}_3$. By Birkhoff's Subdirect Representation Theorem and Theorem A.6, any $\mathbb{S}_3$-algebra can be embedded in such a direct product. Hence any $\mathbb{S}_3$-algebra satisfies $s = s'$. □

Appendix B

Let $\mathcal{D}_4$ be the four-element algebra $\{m, t, f, d\}$ with the operations $\neg$ and $\underset{\circ}{\wedge}$ defined by the corresponding truth tables in Table 7 and 8. By definition, a $\mathbb{D}_4$-algebra will be an algebra with the elements m, t, f and d, and operations $\neg$, $\underset{\circ}{\wedge}$ and $\overset{\circ}{\vee}$ which satisfies the laws (D1)–(D11). $\mathcal{D}_4$ is clearly a $\mathbb{D}_4$-algebra.

We shall argue that the set of equations consisting of (D1)–(D11) completely axiomatizes the equational theory of $\mathcal{D}_4$. Note that these equations are in fact obtained from the axiomatization of $\mathbb{S}_3$, by replacing $*$, $\wedge$ and $\vee$ by m, $\underset{\circ}{\wedge}$ and $\overset{\circ}{\vee}$, respectively, omitting the definition of implication (S5), and adding axiom (D3).

First we shall try to make our claim likely, by considering the reduct $\mathcal{D}$ obtained from $\mathcal{D}_4$ by dropping the constant d. In the algebra $\mathcal{S}_2 \times \mathcal{S}_3$, the operations $\neg$ and $\wedge$ work on the elements $<*,*>, <tf,t>, <tf,f>$ and $<tf,*>$, as follows:

$\neg$	
$<*,*>$	$<*,*>$
$<tf,t>$	$<tf,f>$
$<tf,f>$	$<tf,t>$
$<tf,*>$	$<tf,*>$

$\wedge$	$<*,*>$	$<tf,t>$	$<tf,f>$	$<tf,*>$
$<*,*>$	$<*,*>$	$<*,*>$	$<*,*>$	$<*,*>$
$<tf,t>$	$<*,*>$	$<tf,t>$	$<tf,f>$	$<tf,*>$
$<tf,f>$	$<*,*>$	$<tf,f>$	$<tf,f>$	$<tf,*>$
$<tf,*>$	$<*,*>$	$<tf,*>$	$<tf,*>$	$<tf,*>$

If we read m for $<*,*>$, t for $<tf,t>$, f for $<tf,f>$, and d for $<tf,*>$, these tables are precisely the tables for $\mathcal{D}_4$. So $\mathcal{D}$ is a subdirect product of $\mathcal{S}_2$ and $\mathcal{S}_3$, hence $\mathcal{D}$ and $\{\mathcal{S}_2, \mathcal{S}_3\}$ have the same equational theory, (S1)–(S11) by Appendix A.

Since (S1)–(S11)$\subseteq$(D1)–(D11), lemmas A.1,2 hold for $\mathbb{D}_4$-algebras. On inspection, lemmas A.3–5 will be seen to go through as well. By (D3), in $\mathbb{D}_4$-algebras $d = t$ iff $d = f$. Now the argument of Theorem A.6 shows that the subdirectly irreducible $\mathbb{D}_4$-algebras are precisely the three-element $\mathbb{D}_4$-algebra $\mathcal{D}_3$ with $m = d$, the two-element $\mathbb{D}_4$-algebra $\mathcal{D}_{2,0}$ with $d = t = f$, and the two-element $\mathbb{D}_4$-algebra $\mathcal{D}_{2,1}$ with $t = f$ and $m = d$.

Theorem B.1. (D1)–(D11) completely axiomatize the equational theory of $\mathcal{D}_4$.

Proof. We know that (D1)–(D11) is contained in the equational theory of $\mathcal{D}_4$. The converse follows from the fact that the subdirectly irreducible $\mathbb{D}_4$-algebras are homomorphic images of $\mathcal{D}_4$. □