Bisimulation is two-way simulation

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Abstract

We give here a simple proof of the fact that on transition systems bisimulation is the equivalence relation generated by simulation via functions. The proof entirely rests on simple rules of the calculus of relations.

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Simulation is a standard notion of graph homomorphism that has been used in the study of flow diagram programs (see, e.g. [4, 3, 7]). Bisimulation is an equivalence on transition systems introduced by Park [6] in connection with Milner's work on concurrency [5]. In [2] we have shown that bisimulation is the equivalence relation generated by simulation via functions by using a translation between flowchart schemes and process graphs. We give here a simple proof of this fact using transition systems and simple rules of the calculus of relations, only.

Remarks on notation: We use "$\circ$" to denote relational composition of two binary relations (i.e., $R \circ T = \{(x,z) \mid \exists y: \ x \ R \ y \text{ and } y \ T \ z\}$ and Id_A to denote the identity relation on a set A (i.e., $Id_A = \{(x,x) \mid x \in A\}$).

Definitions: Let A be a set of atomic actions. A *transition system over A with multiple entries and multiple exits* $T : I \xrightarrow{S} O$ consists of three sets of vertices: I, S and O with $S \cap (I \cup O) = \emptyset$ (I specifies the start vertices, S the internal ones and O the end ones) and a transition relation $T \subseteq (I \cup S) \times A \times (O \cup S)$. Two transition systems $T : I \xrightarrow{S} O$ and $T' : I \xrightarrow{S'} O$ are *similar* via a function $\phi : S \to S'$ — written $P \to_\phi P'$ — if

- $(Id_I \cup \phi) \circ T' = T \circ (Id_O \cup \phi)$

T and T' are *bisimilar* via a relation $\rho \subseteq S \times S'$ — written $P \leftrightarrow_\rho P'$ — if

- $(Id_I \cup \rho) \circ T' \subseteq T \circ (Id_O \cup \rho)$
- $(Id_I \cup \rho^{-1}) \circ T \subseteq T' \circ (Id_O \cup \rho^{-1})$

Note that simulation $\to$ is a transitive, but not a symmetric relation, whilst bisimulation $\leftrightarrow$ is an equivalence relation. We denote by $\leftarrow$ the converse of $\to$.

Lemma 1. *For two transition systems $T : I \xrightarrow{S} O$ and $T' : I \xrightarrow{S'} O$ and a function $\phi : S \to S'$ the following equivalence holds.*

$$(i) \quad (Id_I \cup \phi^{-1}) \circ T \subseteq T' \circ (Id_O \cup \phi^{-1}) \quad \Longleftrightarrow \quad (ii) \quad T \circ (Id_O \cup \phi) \subseteq (Id_I \cup \phi) \circ T'$$

Proof: Since ϕ is a function, $\phi \circ \phi^{-1} \supseteq Id_S$ (i.e., ϕ is a total relation) and $\phi^{-1} \circ \phi \subseteq Id_{S'}$ (i.e., ϕ is a univocal relation).

"$\Rightarrow$" Compose (i) on left with $Id_I \cup \phi$ and on right with $Id_O \cup \phi$. Then

$$(Id_I \cup \phi \circ \phi^{-1}) \circ T \circ (Id_O \cup \phi) \subseteq (Id_I \cup \phi) \circ T' \circ (Id_O \cup \phi^{-1} \circ \phi)$$

and this implies (ii) by the above properties of ϕ.

"$\Leftarrow$" Similar. $\square$

Corollary. *Simulation via a function coincides with bisimulation via a function.* $\square$

In order to pass from bisimilar transition systems to chains of similar ones we need a characterization of bisimulation in terms of bisimulation via functions. The result given here is based on a construction of a common refinement of two bisimilar transition systems.

Lemma 2 (interpolation property). *Suppose two transition systems $T' : I \xrightarrow{S'} O$ and $T'' : I \xrightarrow{S''} O$ and a relation $\rho \subseteq S' \times S''$ are given such that $T' \leftrightarrow_\rho T''$. Then there exist a transition system T and two functions $\phi : S \to S'$ and $\psi : S \to S''$ such that $T \leftrightarrow_\phi T'$ and $T \leftrightarrow_\psi T''$.*

Proof: Take a decomposition of ρ as $\phi^{-1} \circ \psi$ for two functions $\phi : S \to S'$ and $\psi : S \to S''$. Bisimulation $T' \leftrightarrow_\rho T''$ means

(i) $(Id_I \ \cup \ \phi^{-1} \circ \psi) \circ T'' \subseteq T' \circ (Id_O \ \cup \ \phi^{-1} \circ \psi)$ and

(ii) $(Id_I \ \cup \ \psi^{-1} \circ \phi) \circ T' \subseteq T'' \circ (Id_O \ \cup \ \psi^{-1} \circ \phi)$

Take the transition system $T : I \xrightarrow{S} O$ given by the transition relation

$$T = [(Id_I \ \cup \ \phi) \circ T' \circ (Id_O \ \cup \ \phi^{-1})] \ \cap \ [(Id_I \ \cup \ \psi) \circ T'' \circ (Id_O \ \cup \ \psi^{-1})]$$

Then $T \leftrightarrow_\phi T'$. Indeed, one inclusion is straighforward:

$$\begin{array}{lll} (Id_I \ \cup \ \phi^{-1}) \circ T & \subseteq & (Id_I \ \cup \ \phi^{-1} \circ \phi) \circ T' \circ (Id_O \ \cup \ \phi^{-1}) \\ & \subseteq & T' \circ (Id_O \ \cup \ \phi^{-1}) \qquad (\phi \text{ is a univocal relation, hence } \phi^{-1} \circ \phi \subseteq Id_{S'}) \end{array}$$

For the second one we use (ii). By a left composition of (ii) with $Id_I \ \cup \ \psi$ we get

$$(Id_I \ \cup \ \psi \circ \psi^{-1} \circ \phi) \circ T' \subseteq (Id_I \ \cup \ \psi) \circ T'' \circ (Id_O \ \cup \ \psi^{-1} \circ \phi)$$

By using $Id_S \subseteq \psi \circ \psi^{-1}$ (i.e., ψ is a total relation) it follows that the left hand side of this inclusion contains $(Id_I \ \cup \ \phi) \circ T'$. On the other hand, the right hand side of this inclusion is equal to the second part of the definition of T composed on right with $Id_O \ \cup \ \phi$, hence is included in $T \circ (Id_O \ \cup \ \phi)$. Consequently,

$$(Id_I \ \cup \ \phi) \circ T' \subseteq T \circ (Id_O \ \cup \ \phi)$$

In a similar way one may prove that $T \leftrightarrow_\psi T''$. $\square$

Theorem. *Bisimulation is the equivalence relation generated by simulation via functions. More precise,* $\leftrightarrow \ = \ \leftarrow \circ \rightarrow$.

Proof: From the lemmas above. □

Simulation is an useful tool to speak about minimization. Simulation via surjective functions models identification of the states with the same behaviour and the *converse* of the simulation via injective functions models the deletion of the nonaccessible parts. We may give an alternative proof of the Theorem by replacing Lemma 2 with another one using minimization (as in [7] or as in Theorem 2.7.13 in [1] regarding normal forms), but the characterization is less direct, i.e., instead of the present characterization $\leftrightarrow \; = \; \leftarrow \circ \rightarrow$ we get $\leftrightarrow \; = \; \rightarrow_{sur} \circ \; {}_{inj}\!\leftarrow \circ \rightarrow_{inj} \circ \; {}_{sur}\!\leftarrow$, where $\rightarrow_{sur}$ and $\rightarrow_{inj}$ are the restrictions of simulation to simulation via surjective and injective functions, respectively. The latter characterization says that two transition systems are bisimilar iff by the identification of the states with the same behaviour and the deletion of the nonaccessible ones both systems may be reduced to the same minimal one.

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